

(MASTER THESIS)

**DISTRIBUTIONS OF THE PARAMETERS IN
VASICEK MODEL**

Gönül AYRANCI

Supervisor: Yrd. Doç. Dr. Banu ÖZGÜREL

Department of Actuarial Science

Bornova-İZMİR

2013

**YASAR UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE**

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This study titled “ Distributions of the Parameters in Vasicek Model” and presented as Master Thesis by Gönül Ayrancı has been evaluated in compliance with the relevant provisions of Y.U Graduate Education and Training Regulation and Y.U Institute of Science Education and Training Direction and jury members written below have decided for the defence of this thesis and it has been declared by consensus / majority of votes that the candidate has succeeded in thesis defence examination dated May 13, 2013.

Jury Members:**Signature:**

Yrd. Doç. Dr. Banu ÖZGÜREL

.....

Yrd. Doç. Dr. Serkan ALBAYRAK

.....

Doç. Dr. Özlem Ege ORUÇ

.....

ÖZET

VASICEK MODELİNDE PARAMETRELERİN DAĞILIMLARI

AYRANCI, Gönül

Yüksek Lisans Tezi, Aktüerya Bilimleri Bölümü

Tez Danışmanı: Assist. Prof. Dr. Banu ÖZGÜREL

Mayıs 2013

Bu çalışmada LIBOR TR faiz oranı verilerinin 2 Ocak 2008 – 5 Aralık 2012 tarihleri arasındaki günlük zaman serilerinin stokastik davranışlarını incelemek için Vasicek faiz oranı modeli kullanılmıştır. Modelde yer alan parametrelerin tahmininde en küçük kareler tahmin yönteminden faydalanılmıştır. Monte Carlo simülasyonu kullanılarak parametrelerin dağılımları incelenmiştir. Bu metod sayesinde sadece parametre tahmini yapılmamış, parametrelere ilişkin güven aralıkları elde edilmiştir.

Anahtar Kelimeler: Vasicek modeli, En Küçük Kareler Yöntemi, Monte Carlo Simülasyonu

ABSTRACT

DISTRIBUTIONS OF THE PARAMETERS IN VASICEK MODEL

AYRANCI, Gönül

MSc in Actuarial Science

Supervisor: Assist. Prof. Dr. Banu ÖZGÜREL

Mayıs 2013

In this study time series of TRLIBOR interest rates in different maturities are modeled with Vasicek Model. OLS method is used for calibrating the model parameters of Vasicek to TRLIBOR rates data which is between 2.01.2008 and 5.12.2012. Then distributions of parameters are obtained by using Monte Carlo Simulation regarding the Vasicek Model. Thus, not only parameters are estimated but also confidence intervals are given.

Keywords: Vasicek Model, Ordinary Least Square Method, Monte Carlo Simulation

ACKNOWLEDGEMENTS

I would not have been able to complete this journey without the aid and support of countless people.

First I must express my gratitude to my supervisor Assist. Prof. Dr. Banu Özgürel for the guidance, useful comments, remarks and engagement through the learning process of this master thesis.

I want to express the separate gratitude to Assist. Prof. Dr. Serkan Albayrak for his advices, irreplaceable contributions, support and especially for his helps in application of this study.

I am also grateful to Prof. Dr. Behzat Gürkan for encouragement and support in many ways to the success of this study.

Last but not least, I would like to thank my parents and friends, especially my Mother and Father for their unconditional support throughout my study.

TEXT OF OATH

I declare and honestly confirm that my study titled “Distributions of Parameters in Vasicek Model”, and presented as Master’s Doctorate Thesis has been written without applying to any assistance inconsistent with scientific ethics and traditions and all sources I have benefited from are listed in bibliography and I have benefited from these sources by means of making references.

13/ 05/ 2013

Gönül AYRANCI

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INDEX OF SYMBOLS AND ABBREVIATIONS

<u>Abbreviations</u>	<u>Explanation</u>
CIR	Cox-Ingersoll-Ross
MLE	Maximum Likelihood Estimation
OLS	Ordinary Least Squares
SCR	Solvency Capital Requirement
SDE	Stochastic Differential Equation

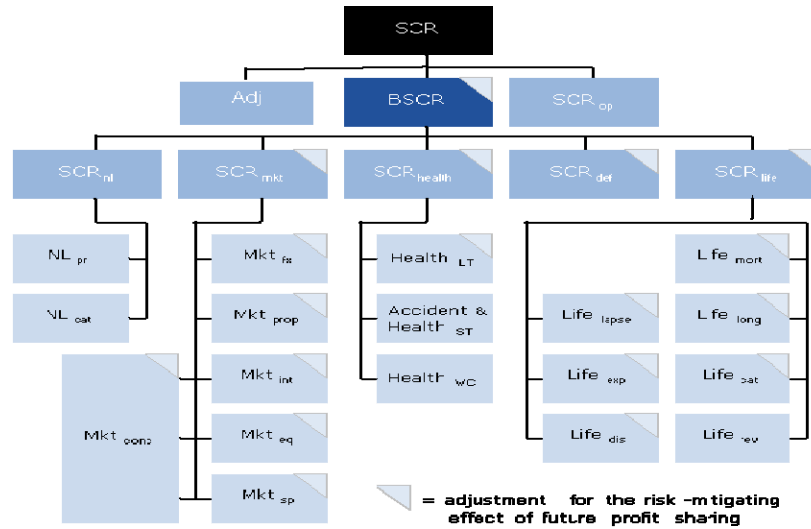
1 INTRODUCTION

Interest rates are an indicator for the global cost of the money. There are financial instruments which directly link with interest rates in portfolio of financial institutions and especially of insurance companies. Interest rates and the volatility of interest rates play a crucial role in estimation of losses which is caused for holding the money. Moreover there are some government regulations about estimation of those losses for insurance companies (**Solvency II**). Insurance companies have to calculate the capital requirement for the market risk under the Solvency II requirements. There is a committee that is established by Treasury on 2009 for inform the sector about Solvency II directives and assess the impact of these directives on the insurance sector. Ten insurance companies, of which 5 operate in life, 4 non-life and 1 in reinsurance, participated in these studies. The Solvency Capital Requirement (SCR) is the new solvency standard for firms in Solvency II directives. SCR is based on the idea that an insurance firm should have the amount of capital that is sufficient with a 99.5% confidence-level to guarantee that the firm will have enough assets to cover its liabilities at the end of one year period. In other words, SCR calibrated to the Value-at-Risk of basic own funds subject to a confidence level 99.5% over a 1 year time horizon. The SCR is intended to reflect all quantifiable risks that the firm might face. A basic SCR calculation divided into modules per risks, with adjustment for the loss-absorbing capacity of technical provisions and deferred taxes. These calculations must be done at least once per year using a standard formula or an internal model. But internal models improve risk sensitivity of SCR and provide better risk management, which also improves policyholder protection (CP20 final advice¹). Martin (2012) applies the model of Gatzert and Martin (2012) for deriving a partial internal model for the market risk module regarding the underlying interest rate process and parameters. Solvency II Directive gives an option to model only certain risk modules or business units and to use the standard formula for the remaining parts.

¹ CP20 is a consultation process initiated by CEIOPS (Committee of European Insurance and Occupational Pensions Supervisors) on the “advice to the European Commission in the Framework of the Solvency II”

There is a figure below that shows the risk modules of SCR (Figure 1.1).

Figure 1.1 Risk Modules of SCR



As we can see in the Figure 1.1, interest rate risk plays a significant role for calculating SCR (Martin, 2012). The aim of this study is to evaluate model with focus on market risk module of Solvency II regarding the interest rate risk. Already it is an inevitable fact that insurance companies must have fitting interest rate model while they estimate the value at risk (VaR) on a specific portfolio of financial assets. Also Martin (2012) shows that the change in SCR is sustainable when changing the underlying interest rate model.

Modeling of interest rates is especially used for investment and financial decisions, portfolio management and insurance. Future interest rates can be predicted with a suitable interest rate model and those interest rates predictions can be used for the estimation of price of the financial instruments. An increasing number of insurance companies and other financial institutions are using stochastically generated interest rates to measure interest rate risk and to value financial assets. The use of stochastic interest rates for insurance system is both appropriate— given the stochastic nature of interest rates— and increasingly mandated by regulators at the larger more complex financial institutions. An interest rate model is a probabilistic description of the future evolution of interest rates. It characterizes the uncertainty involved in interest rates. Interest rate modeling is a very important part of any insurance. This is because the valuation of the assets and liabilities of insurance companies depends crucially on interest

rates. Also enormous literature in finance such as Vasicek (1977) and Cox, Ingersoll and Ross (CIR) (1985) have documented that interest rate should be followed by a stochastic process.

In the literature various models have been proposed such as Merton (1973), Vasicek (1977), Dothan (1978), Richard (1979), Brennan and Schwarz (1979, 1982), Cox, Ingersoll and Ross (1985), Rendleman and Bartter (1980), Longstaff (1989), Hull and White (1990), Pearson and Sun (1994). These models can be used to value all interest rate contingent claims in different cases. Major difficulty in the modeling process is to provide parameters that fit to the initial term structure of interest rates. In that case several papers focused on the estimation of the parameters in a consistent way. Chan et al. (1992) estimate and compare continuous time models of the short term riskless rate using the Generalized Method of Moments. Kladvikova (2007) investigates Maximum Likelihood Estimation of the CIR process for interest rate time series. While comparing the Vasicek and CIR models, Munnik and Schotman (1992) used Ordinary Least Square Estimation method. Berg (2011) describe OLS and MLE methods calibrating the model parameters of Vasicek. Also Zeytun and Gupta (2007) analyze Vasicek and CIR models to give results for the parameters. Carriere (2000) investigate long-term yield rates for valuing a life insurance and used OLS method parameter estimation of the models such as Vasicek (1977), CIR (1985), Nelson-Siegel (1987). Önalın (2009) explains the structure of Vasicek and CIR models, then give parameter estimation methods OLS for Vasicek, Matingale method for CIR. Moreover Şahin and Genç (2009) investigate which short term interest rate model appropriate for Turkish data.

In this paper we examine the application of the Vasicek Model with TRLIBOR data to study the term structure of interest rates. Simply the aim of this study is to determine parameters of the Vasicek model and give confidence intervals regarding to estimation of parameters. After model parameters are estimated by using OLS method, the distribution of these estimated parameters is obtained by using the Monte Carlo Simulation.

The rest of this thesis is organized as follows. Chapter 2 describes our model and methods that we used. This chapter is explained in three main sections. First general diffusion model description for interest rates is given and Vasicek Model examined in detail. Then OLS and MLE methods are applied for the parameters of the Vasicek Model. Last section in chapter 2 Monte Carlo method is explained with examples. Chapter 3 shows an application of parameter estimation of Vasicek Model for TRLIBOR rates. Chapter 4 summarizes the paper and gives results that we obtain.

2 METHODS

2.1 Model Description

Interest rate processes have been defined with stochastic differential equations. Generally models divided into equilibrium models and no-arbitrage models. We only focus on equilibrium models and from those only the Vasicek model describe here. The equilibrium models based on some assumptions about economic variables and give us process for the short rate, r . (Hull and White 1993). To achieve a balance between supply of bonds rate and demand of the investors is the aim of the equilibrium models. The most well-known equilibrium models are Merton(1973), Vasicek(1977) and CIR(1985).

There are also single or multi-factor models for the interest rate modeling and the Vasicek model is classified as one-factor model and the interest rate is called instantaneous interest rate. In one-factor models it is accepted that there is only one source of the risk. These models may be preliminary among the recent multi-factor models but they provide a good introduction to the study of interest rates. Moreover they can be easily solved and Sorwar (2007) states that the existence of analytical solutions leads to quick valuation of the bond, option, prices, and the hedge parameters such as delta for risk management purposes.

The general formula for one-factor diffusion model is:

$$dr = \mu(r, t)dt + \sigma(r, t)dW \quad (1)$$

where dW is a Brownian motion process. Different forms of functions μ and σ give various diffusion models.

2.1.1 Vasicek Interest Rate Model

Vasicek model is a one-factor model describing an evolution of interest rates. It describes movements of interest rates as driven by source of market risk. Also it can be used for the derivatives of interest rates. It was introduced in 1977 by Oldrich Vasicek. The spot rate $r(t)$ follows so-called Ornstein-Uhlenbeck process and the equation has following form

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t) \quad (2)$$

$$r(0) = r_0 \quad (3)$$

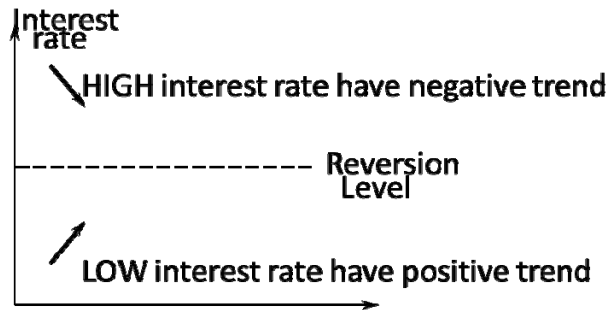
where W_t is a Wiener process (also called Brownian Motion) under the risk neutral measure modelling the random market risk factor. With $\alpha > 0$, corresponding to the choice $\mu(t, r) = \alpha(\beta - r)$, $\sigma(t, r) = \sigma$ in equation (1). This description of the spot rate process has been proposed by Merton (1971).

According to the Vasicek O. (1977), Journal of Financial Economics, 5, 177-188, "The Ornstein-Uhlenbeck process with $\alpha > 0$ is sometimes called the elastic random walk. It is a Markov process with normally distributed increments. In contrast to the random walk (the Wiener process), which is an unstable process and after long time will diverge to infinite values, the Ornstein-Uhlenbeck process possesses a stationary distribution. The instantaneous drift $\alpha(\beta - r)$ represents a force that keeps pulling the process towards its long-term mean β with magnitude proportional to the deviation of the process from the mean. The stochastic element, which has a constant instantaneous variance, causes the process fluctuate around the level β in an erratic, but continuous, fashion"

The most important feature which this model exhibits and explained by Vasicek is the mean reversion, which means that if the interest rate is bigger than the long run mean ($r > \beta$) then the coefficient $\alpha(> 0)$ makes the drift become negative so that the rate will be pulled down in the direction of β . Similarly, if the interest rate is smaller than the long run mean ($r < \beta$), then the coefficient $\alpha(> 0)$ makes the drift term become positive so that the rate will be pulled up in

the direction of β . Therefore, the coefficient α is the speed of adjustment of the interest rate towards its long run level.

Figure 2.1 Mean reversion process



One of the most attractive feature of the Vasicek Model is that there is close form solution for r_t . We start solution of the Stochastic Differential Equation (2) by taking the derivative of $e^{\alpha t} r_t$, which yields:

$$d(e^{\alpha t} r_t) = r_t \alpha e^{\alpha t} dt + e^{\alpha t} dr_t \quad (4)$$

Substitute equation (2) into equation (4) yields;

$$d(e^{\alpha t} r_t) = r_t \alpha e^{\alpha t} dt + e^{\alpha t} [\alpha(\beta - r_t)dt + \sigma dW_t] \quad (5)$$

Rearrange the equation (5) gives:

$$d(e^{\alpha t} r_t) = e^{\alpha t} [\alpha\beta dt + \sigma dW_t] \quad (6)$$

Taking integral from s to t for both sides of the equation (6):

$$\int_s^t d(e^{\alpha u} r_u) = \int_s^t e^{\alpha u} [\alpha\beta du + \sigma dW_u] \quad (7)$$

$$e^{\alpha t} r_t - e^{\alpha s} r_s = \alpha \beta \int_s^t e^{\alpha u} du + \sigma \int_s^t e^{\alpha u} dW_u \quad (8)$$

$$e^{\alpha t} r_t - e^{\alpha s} r_s + \beta(e^{\alpha t} - e^{\alpha s}) + \sigma \int_s^t e^{\alpha(t-u)} dW_u \quad (9)$$

If we multiply both sides of the equation with $e^{-\alpha t}$ the solution of the differential equation (2) is:

$$r_t = e^{-\alpha(t-s)} r_s + \beta(1 - e^{-\alpha(t-s)}) + \sigma \int_s^t e^{-\alpha(t-u)} dW_u \quad (10)$$

r_t process given in equation (10) follows a Gaussian Distribution with mean:²

$$E[r_t] = \beta + (r_s - \beta)e^{-\alpha(t-s)} \quad (11)$$

and variance:

$$\text{Var}(r_t) = \frac{\sigma^2}{2\alpha(1 - e^{-2\alpha(t-s)})} \quad (12)$$

If time increases the mean tends to the long-term value β and the variance remains bounded, implying mean reversion. The long-term distribution of the Ornstein-Uhlenbeck process is stationary and is Gaussian with mean β and

variance . (Sypkens 2010)

² Önalın Ö., Vasicek ve CIR Modelleri Kullanılarak Oynaklık ve Faiz Oranlarının Modellenmesi, Marmara Üniversitesi, İİBF dergisi, II, S. 329-344, 2009.

2.2 Parameter Estimation

2.2.1 Ordinary Least Square Estimation for Vasicek Model

The explicit solution of the Ornstein-Uhlenbeck SDE is obtained in previous section, namely:

$$r_t = e^{-\alpha(t-s)}r_s + \beta(1 - e^{-\alpha(t-s)}) + \sigma \int_s^t e^{-\alpha(t-u)} dW_u \quad (13)$$

r_t is a random variable with mean and variance $e^{-\alpha(t-s)}r_s + \beta(1 - e^{-\alpha(t-s)})$ and $\frac{\sigma^2}{2\alpha(1 - e^{-2\alpha(t-s)})}$ respectively. It is assumed that r_t follows a **AR(1)** process as follow:

$$r_{t_i} = a + br_{t_{i-1}} + \delta \varepsilon_{t_i} \quad (14)$$

The discrete version of the equation (13) on time grid

(equal time intervals) and $\Delta t = t_i - t_{i-1}$

$$\Delta r_{t_i} = \alpha(\beta - r_{t_{i-1}})\Delta t + \sigma \Delta \varepsilon_{t_i} \quad (15)$$

where ε_{t_i} is a Gaussian white noise ($\varepsilon \sim N(0,1)$).

Parameters of equation (14) is:

$$a = \beta(1 - e^{-\alpha\Delta t}) \quad (16)$$

$$b = e^{-\alpha\Delta t} \quad (17)$$

$$\delta = \sigma \sqrt{\frac{1 - e^{-2\alpha\Delta t}}{2\alpha}} \quad (18)$$

Rearrange equations (16), (17) and (18) in terms of the parameters α , β , and σ which yield:

$$\beta = \frac{a}{1-b}, \quad \alpha = -\frac{\ln b}{\Delta t}, \quad \sigma = \delta \sqrt{\frac{-2 \ln(b)}{\Delta t(1-b^2)}} \quad (19)$$

The formulas for the least squares estimates a , b , and ε_{t_i} are:

$$\hat{a} = \frac{\sum_{i=1}^n r_{t_i} - b \sum_{i=1}^n r_{t_{i-1}}}{n} \quad (20)$$

$$\hat{b} = \frac{n \sum_{i=1}^n r_{t_{i-1}} r_{t_i} - \sum_{i=1}^n r_{t_{i-1}} \sum_{i=1}^n r_{t_i}}{n \sum_{i=1}^n r_{t_{i-1}}^2 - \left(\sum_{i=1}^n r_{t_{i-1}} \right)^2} \quad (21)$$

$$\hat{\varepsilon}_{t_i} = \sqrt{n \sum_{i=1}^n r_{t_i}^2 - \left(\sum_{i=1}^n r_{t_i} \right)^2} \quad (22)$$

2.2.2 Maksimum Likelihood Estimation for The Vasicek Model

MLE method is consistent for the Vasicek model since its error term is normally distributed (Ren Raw Chen, 1996). So, the likelihood function can be derived for the MLE. We need to define the likelihood function and find the parameters that maximize the likelihood function. There is different methods in literature for maximize the methods likelihood function (Sykens, 2010). Optimization method is explained for this problem. The conditional probability function is derived by combining the solution of our model with the normal distribution function (Calibrating the Vasicek Thijs Von den Berg, 2011).

$$P(N = x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad (23)$$

If $r_{t_{i-1}}$ is given, the conditional density function of r_{t_i} on the time grid $t_1, t_2, t_3, \dots, t_n$ with equal time step $\Delta t = t_i - t_{i-1}$ by using equation (11) and (12).

$$f(r_{t_i}, \alpha, \beta, \sigma) = (2\pi)^{-\frac{1}{2}} \left[\frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}) \right]^{-\frac{1}{2}} \exp \left[\frac{-\left(r_{t_i} - \beta - \left(r_{t_i} - \beta - (r_{t_{i-1}} - \beta) e^{-\alpha(t_i - t_{i-1})} \right)^2 \right)}{2 \left(\frac{\sigma^2}{2\alpha} \right) (1 - e^{-2\alpha \Delta t})} \right] \quad (24)$$

Since the logarithmic function is monotonically increasing, maximizing the log-likelihood function also maximizes the likelihood function. The log-likelihood function is;

$$\begin{aligned} \ln L(\theta; \beta, \alpha, \sigma) &= -\frac{n}{2} \ln \left(\frac{\sigma^2}{2\alpha} \right) - \frac{1}{2} \sum_{i=1}^n \ln(1 - e^{-2\alpha \Delta t}) \\ &= -\frac{n}{2} \sum_{i=1}^n \frac{\left(r_{t_i} - \beta - (r_{t_{i-1}} - \beta) e^{-\alpha \Delta t} \right)^2}{1 - e^{-2\alpha \Delta t}} \end{aligned} \quad (25)$$

We can find the maximum likelihood estimates $\hat{\theta}$ of parameter vector θ by maximizing maximizing the log likelihood function (Equation 25):

$$\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma}) = \underset{\theta}{\operatorname{argmax}} \ln L(\theta) \quad (26)$$

2.3 Monte Carlo Simulation

Monte Carlo means using random numbers in scientific computing. More precisely, it means using random numbers as a tool to compute something that is not random. One of the first uses of this method is suggested by Buffon (1777) with an experiment that calculates the probability that the needle intersect one of the lines on a horizontal plane, now known as the famous Buffon's Needle Problem. Later Laplace (1886) suggested the idea of evaluating π with Monte

Carlo Simulation. Also it is used in kinetic theory of gases by Kelvin (1901). But the method today labeled as “Monte Carlo Method” is formally introduced to the literature by Ulam (1940) where he advocated the use of computers in an integration problem related to Manhattan Project which has no formal solution.

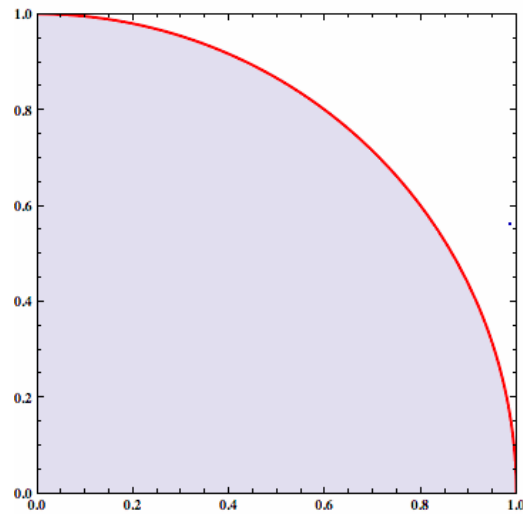
The process of method can be explained as follows: During a Monte Carlo Simulation, values are chosen randomly from the input probability distributions. These generated samples are called iterations and results from the samples are recorded for each iteration. Through Monte Carlo simulation, this process can be done hundreds and thousands times. In this way, Monte Carlo simulation provides much more extensive information about what may happen and how likely it is to happen.

We can explain Monte Carlo method further with a nice and well-known example of π value estimation. π is a constant that is the ratio of a circle’s circumference to its diameter. It is simply accepted as 3.14 but true value cannot represent as a finite decimal fraction. Let us consider a unit circle with a radius 1. If (x, y) is a point on the unit circle, by the Pythagorean Theorem, that satisfy the following equation;

$$x^2 + y^2 = 1 \quad (27)$$

By choosing random points on the plane, we need to decide each point that we have chosen whether is within the circle or not. The quarter unit circle is considered as shown in the following figure.

Figure 2.2 Quarter circle of radius r=1



Since we know the area of a circle with radius r .

$$A = \pi r^2 \quad (28)$$

The unit circle area is equal to π and the area of the quarter circle that we interested is equal to $\frac{\pi}{4}$. We have to look at the unit circle (radius = 1) within a square with sides equal to 1 (see Figure 2.2). Now if we pick a random points (x, y) where both x and y are between from 0 to 1. The probability of that this random point lies inside the unit circle is given as the proportion between the area of the unit circle and the square :

$$P = \frac{\text{Area of the circle}}{\text{Area of the square}} = \frac{\pi}{4} \quad (29)$$

If we pick a random point n times and m of those times the point lies inside the unit circle, the probability of that a random point lies inside the unit circle is given as :

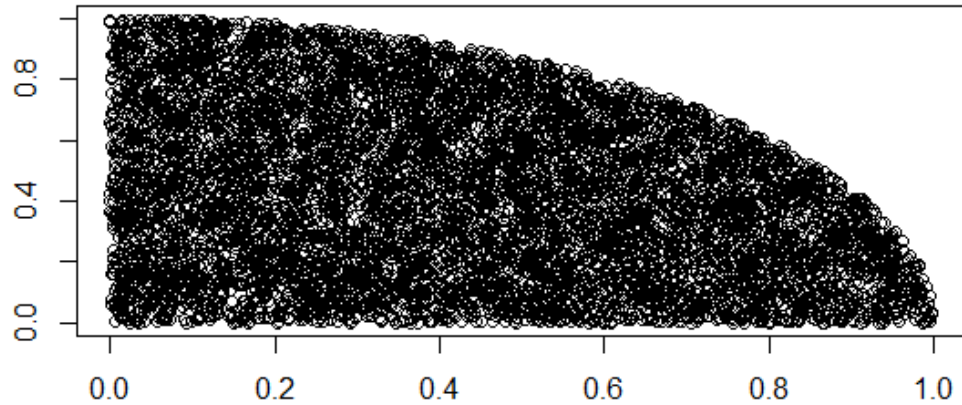
$$P(x^2 + y^2 < 1) = \frac{m}{n} \quad (30)$$

where the dot indicates that this is a discrete distribution (because M and N are integers). But if N becomes very large (theoretically infinite), the two probabilities will become equal and we can write :

$$\pi = \frac{4M}{N} \quad (31)$$

Monte Carlo method applied and simulation run 10000 times to approximating the value of π . Obtaining results and the codes relating to the process are given (Figure 2.3).

```
n=10000
x=runif(n)
y=runif(n)
counter=0
plot(NA,type="n",xlim=c(0,1),ylim=c(0,1),
xlab="n=10000",ylab="")
for (i in 1:n){
  if ((x[i]^2+y[i]^2)<=1) {
    counter=counter+1
    points(x[i],y[i],type="p")
  }
}
4*counter/n
```

Figure 2.3 π value estimation with Monte Carlo Method

$$n = 10000 \quad (\pi \approx 3.142)$$

Monte Carlo simulation can be used to approximate probability distributions of functions of random variables. For example, let X be a random variable and its expected value be $E[X]$. If we generate X_1, X_2, \dots, X_n independent random variables from the same distribution then we can approximate $E[X]$ by

$$\bar{X} \approx \frac{1}{n} \sum_{k=1}^n X_k \quad (32)$$

The strong law of large numbers states that $\bar{X} \rightarrow E[X]$ as $n \rightarrow \infty$. Albeit the generated random numbers X_k would be different each time and therefore so does the sample average, \bar{X} , the target number, $E[X]$, is fixed and nonrandom. Using thousands of Monte Carlo runs one can approximate the distribution of \bar{X} , around $E[X]$.

Monte Carlo Methods allows us to account for risk in quantitative analysis and decision making. The technique is used in various fields such as finance (Haugh, 2004), project management (Kwak and Ingall, 2007), manufacturing (Pica et al., (2006), engineering (Amar, 2006), insurance (Collins, 1962), the environmental and resource economics (Scarpa and Alberini, 2005).

3 APPLICATION

In this thesis, we are going to use TRLIBOR data with different time to maturities. The Turkish Lira Interbank Offered Rate, or TRLIBOR, is the average interest rate at which term deposits are offered between prime banks in the Turkish wholesale money market or interbank market.

TRLIBOR initiated on August 1, 2002 by the Banks Association of Turkey and has resulted in the formation of benchmark interest rates that the economic agents base their projections. The formation of benchmark interest rates is an important development as they will increase transparency of the markets, form a sound yield curve in money markets and improve forward transactions by pricing financial assets more effectively.

Our study sample from the web page of the Banks Association of Turkey TRLIBOR³ data period is from 2.01.2008 to 5.12.2012 and contains 1230 daily observations for each time maturity. The rates are shown in the following figures in different maturities, 1 month (Figure 3.1), 6 month (Figure 3.2), 1 year (Figure 3.3) and overnight (Figure 3.4).

Figure 3.1 Time series of TRLIBOR 1M from 2.01.2008 to 5.12.2012

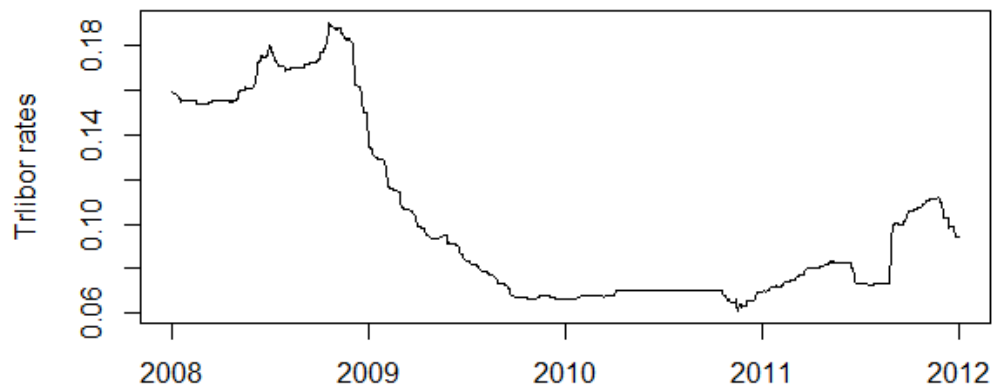
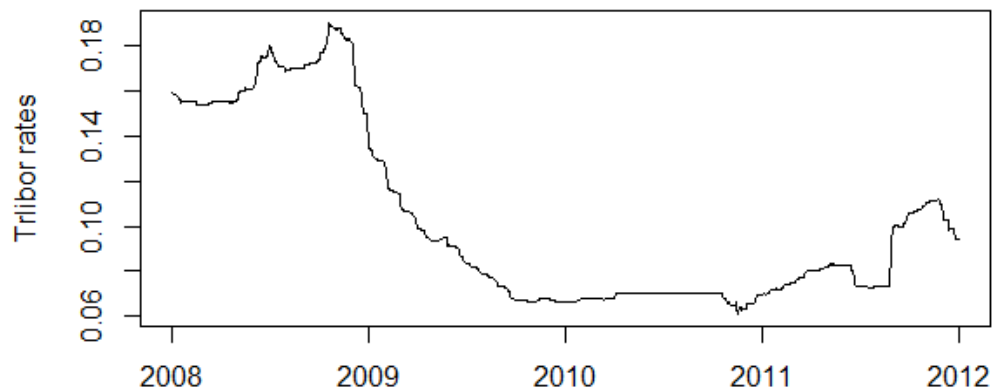
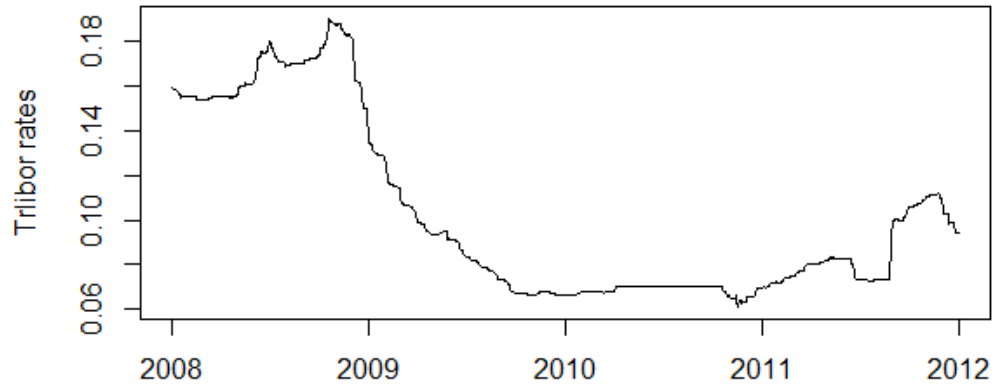


Figure 3.2 Time series of TRLIBOR 6M from 2.01.2008 to 5.12.2012



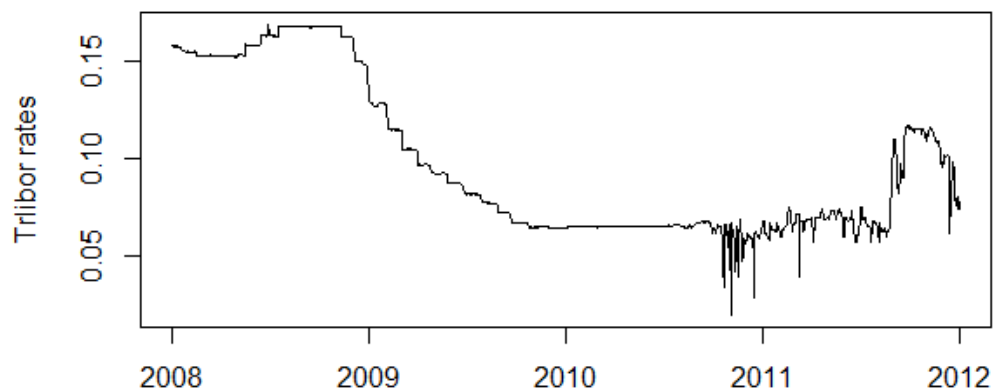
³ <http://www.tbb.org.tr>

Figure 3.3 Time series of TRLIBOR 1Y from 2.01.2008 to 5.12.2012



It can clearly be seen from the figures (Figure 3.1, Figure 3.2 and Figure 3.3) in 1 month, six month and 1 year maturities same pattern is observed. TRLIBOR rates were extremely high at the start of monitored period, especially in last months of 2008. After that there is a dramatic fall occurred in earlier months of 2009 and then it seems stable from 2010 to 2011. Finally, there is an increasing trend between the year 2011 and 2012.

Figure 3.4 Time series of TRLIBOR overnight from 2.01.2008 to 5.12.2012



In the case of overnight maturity, we observed a pattern that more sensitive than the others especially on the last interval from 2011 to 2012. So to estimate the parameters of Vasicek Model we use the daily TRLIBOR data with overnight maturity.

The time series plots of our sample data is shown in the figures above. Since we can observe from the figures, modeling interest rates is a very complex and hard task. So Vasicek Model parameters will be estimated using TRLIBOR data.

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t) \quad (33)$$

$$r(0) = r_0 \quad (34)$$

where α : speed of mean reversion β : long run mean

σ : instantaneous volatility $W(t)$: Weiner process

In order to estimate the development of the instantaneous interest rate the Euler-discretisation of the Vasicek Model (Equation 33) is used and is given:

$$\Delta r = \alpha(\beta - r)\Delta t + \sigma \varepsilon_t \sqrt{\Delta t} \quad (35)$$

where ε_t is generated random number from $N(0,1)$ (Josef, 2009).

In the simulation process, the most important step is to estimate the parameters of our model α, β and σ accurately. Parameter estimation methods regarding the Vasicek Model are described in Chapter 2. Estimation of parameters with OLS method gives very similiar results to MLE (Sypkens, 2010). To estimate the parameters of our model we use OLS method and run the R code below which estimates required parameters (Table 3.1)

```
Vasicek.OLS = function(data,dt){
  N = length(data)
  rate = data[2:N]
  lagrate= data[1:(N-1)]
  a=(N*sum(rate*lagrate)-sum(rate)*sum(lagrate))/
    (N*sum(lagrate^2)- sum(lagrate)^2)
  alphahat = -log(a)/dt
  betahat = sum(rate-a*lagrate ) / (N*(1-a))
}
```

```

v2hat=sum((rate-lagrate*alpha-betahat*(1-a))^2)/N
sigmahat=sqrt(2*alpha_hat*v2hat/(1-a^2))
c(betahat,alphahat,sigmahat)}

```

Table 3.1 Summary of Vasicek Estimators

	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\sigma}$
OLS	0.0799080219842794	1.55095522339525	0.0635675446843325

To estimate distribution of the parameters that fits Vasicek Model, Monte Carlo Simulation explained in Chapter 2 is used and the codes of process are given:

```

init.value=test.data[sizedata[1]]
sizeofsim=1000
sizeofMonteCarlo=1000
delta.t=1/260
set.seed(1)
TEZ=matrix(data=NA,nrow=sizeofMonteCarlo,ncol=3)
for (j in 1:sizeofsim){
  r=init.value
  for (i in 2:sizeofsim){
    delta.r=(outt[2]*(outt[1]-r[(i-
1)])*delta.t)+(outt[3]*rnorm(1)*sqrt(delta.t))
    tempo=r[(i-1)]+delta.r
    r=c(r,tempo)
  }
  TEZ[j,]=Vasicek.OLS(data=r,dt=1/260)
}

```

In each run of the Monte Carlo Simulation process, new parameter values are estimated from the interest rate observations. Each run initiates from a different

random point. Furthermore at each step randomized **variates** enter the model distinguishing each run. Monte Carlo does this process thousands of times. This process runs with two simulation steps. The histograms and plots of the simulation results are given.

Figure 3.5 Plots of the parameters (β, α, σ) in $n = 1000$ simulation step

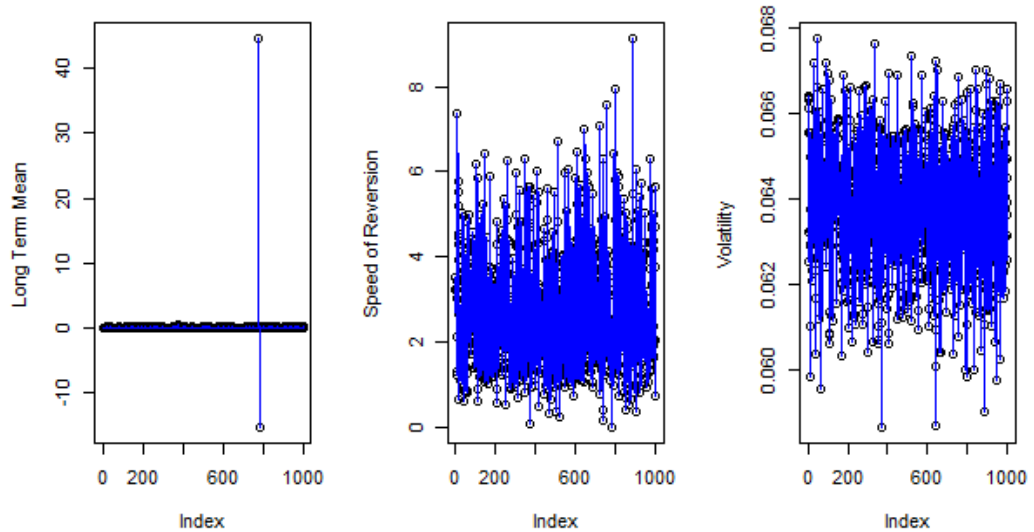
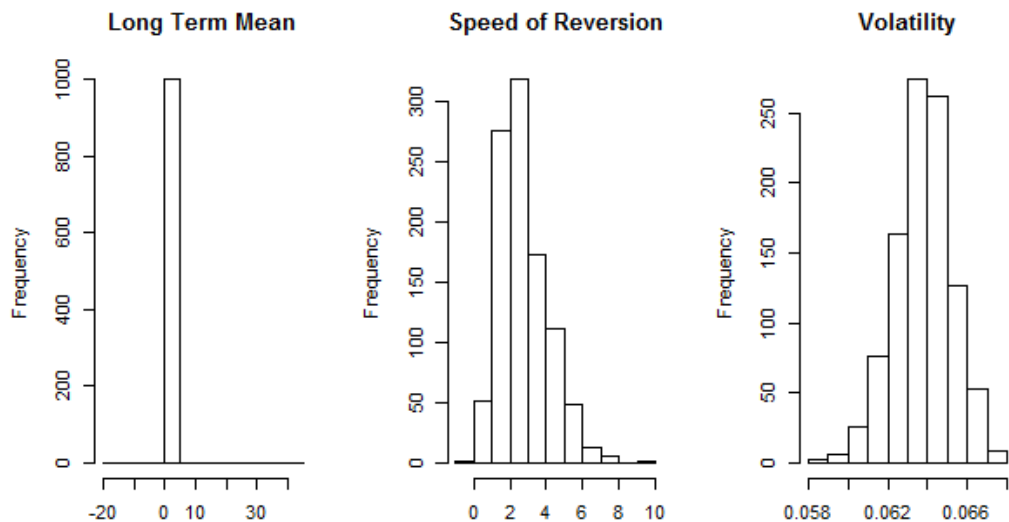


Figure 3.6 Histograms of the parameters (β, α, σ) in $n = 1000$ simulation step



As we can see in the Figure 3.5 observations of estimation, which of long run mean from simulated values of interest rate, have extreme values (outliers) relative to other observations. Ghosh and Vogt (2012) states that there is three main approaches to eliminate outlier problems:

1. To keep the outlier and treat it like any other data point
2. To modify its value
3. To eliminate it

Also they state that method 2 and 3 introduce statistical bias and may undervalue the outlier.

To obtain distribution of parameters, increase our sample size the simulation step is increased to 10000 times.

Figure 3.7 Plots of the parameters (β, α, σ) in $n = 10000$ simulation step

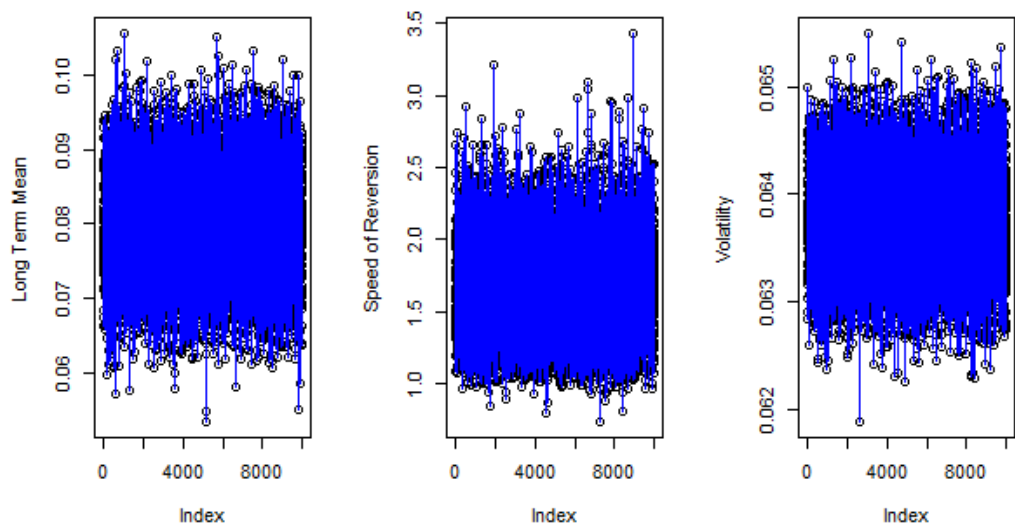
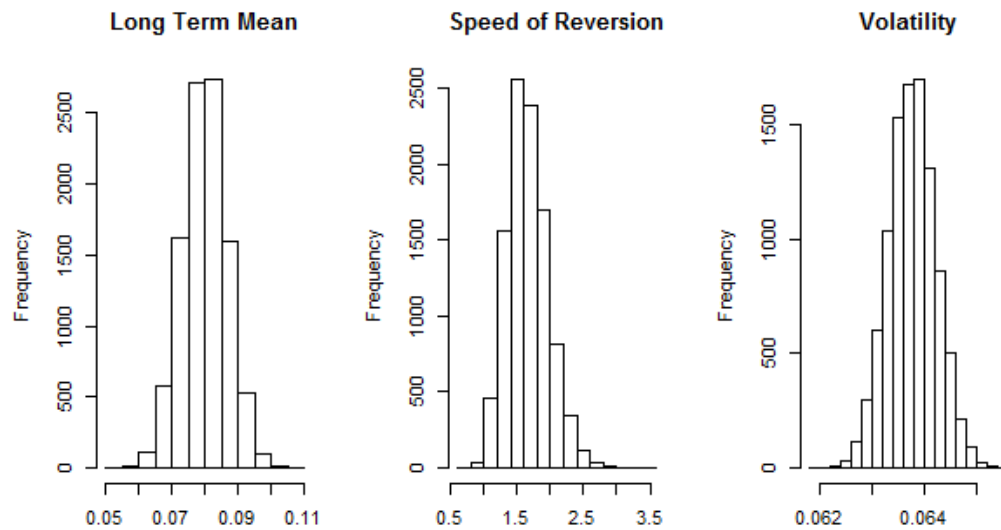


Figure 3.8 Histograms of the parameters (β, α, σ) in $n = 10000$ simulation step

Now after the simulation with time step $n=10000$, parameters are normally distributed as it can be observed in the Figure 3.7 and Figure 3.8. We give our results in Table 3.2 %95 confidence intervals and Table 3.3 %99 confidence intervals for the estimated parameters.

Table 3.2 %95 Confidence Intervals for the parameters

Parameters	%95 Confidence Interval	
Long Term Mean β	0.06697832	0.09310074
Speed of Reversion α	1.126175	2.326765
Volatility σ	0.06288429	0.06465596

Table 3.3 %99 Confidence Intervals for the parameters

Parameters	%99 Confidence Interval	
Long Term Mean β	0.06288056	0.09721137
Speed of Reversion α	1.015408	2.585334
Volatility σ	0.06265065	0.06454089

As we can see from Table 3.2 , there is a significant difference between the confidence interval borders for the long term mean and speed of reversion parameters. Also increasing confidence level (%99) caused increase in interval length. Any chosen parameter ignores the confidence intervals may cause over estimation or under estimation of the model parameters and interest rates.

4 CONCLUSION

This study has been focuses on the distribution of the Vasicek Model parameters. First we emphasized the importance of interest rate modeling especially in insurance sector. As we indicate in introduction, with the new regulations about capital requirements (Solvency II) modeling interest rates arise one of the most important problem in that field. First we gave a general form of the stochastic differential equation for the interest rate modeling. The focus is on the Vasicek Model. The solution of the stochastic differential equation of Vasicek Model is given and it is shown that two different parameter estimation methods,

Ordinary Least Square and Maximum Likelihood estimation, can be applied to the model. Then Monte Carlo Simulation Method is explained.

In the third part of this study is the application of the model on a data set from TRLIBOR rates. We simulated new data set that is approximated by Euler discretization with parameters from OLS estimators. This process applied with Monte Carlo Method for 1000 time steps and 10000 time steps. First simulation gave us distribution of the long term mean with outliers and also high variance. In order to achieve greater accuracy, we increased the sample size to 10000. After we clearly saw that our estimated parameter values from simulations are normally distributed, we gave our results with the %95 and %99 confidence intervals for the parameters.

In this study our aim is observing the distribution of the Vasicek Model parameters. There are already various studies about parameter estimation of the Vasicek and other diffusion models. But we used Monte Carlo Method for obtaining the distribution of the estimated parameters. Through this technique we observed probability distributions of parameters and we also got confidence intervals. It can be clearly seen major difference between the borders of the intervals for long term mean and mean reversion parameters. So during the parameter estimation process confidence intervals are necessary for more accuracy. This step is the key element of our study because probability distributions are much more realistic way describing uncertainty in variables of risk analysis.

5 BIBLIOGRAPHY

Berg T., Calibrating the Ornstein-Uhlenbeck (Vasicek) Model,2011

Sorwar, G., Barone-Adesi, G. and Allegretto, W., (2007)., "Valuation of Derivatives Based on Single-Factor Interest Rate Models", *Global finance journal*, Vol.18

Brigo, D., Neugebauer, M., Dalessandro, A., and Triki, F., (2009). A stochastic Processes Toolkit for Risk Management, Journal of Risk Management in Financial Institutions, Vol. 3, 1.

Hsiao C. and Semmler W., Modeling Short Term Interest Rates, (2002), Elsevier Science

Josef O. (2009), Models of Interest Rate Evolution-Vasicek and CIR Models, Journal of Applied Mathematics, volume II

Önalın Ö., (2009) Vasicek ve CIR Modelleri Kullanılarak Oynaklık ve Faiz Oranlarının Modellenmesi, Marmara Üniversitesi, İ.İ.B.F. Dergisi, Sayı II, S. 329-344.

Şahin H. and Genç İ., (2009), Kısa Dönem Faiz Modellerinin Türkiye için Ampirik Analizi, BDDK Bankacılık ve Finansal Piyasalar, Cilt 3, Sayı 2.

Vasicek O., (1977), An equilibrium characterization of the term structure, Journal of Financial Economics, 5, 177.186.

Benninga S. and Wiener Z., (1998), Term Structure of Interest Rates, Mathematica in Education and Research, Vol. 7 No. 2

Martin M., (2012), Assessing the Model Risk with Respect to the Interest Rate Term Structure under Solvency II, FAU.

Zeytun S. and Gupta A., (2007), A Comparative Study of the Vasicek and the CIR Model of the Short Rate, Berichte des Fraunhofer ITWM, Nr. 124

Hull J. and White A., (1990), Pricing Interest Rate Derivative Securities, The Review of Financial Studies, Volume 3, number 4, p. 573-392.

Kalos H. and Whitlock P., (2008), Monte Carlo Methods, Wiley-Blackwell

Tang C. and Chenb S., (2009), Parameter Estimation and Bias Correction for Difusion Processes., Journal of Econometrics 149, (2009) 65-81.

Acar O., (2009), Avrupa Birliđi Solvency II Projesi, Türkiye Sigorta ve Reasürans Şirketler Birliđi.

Giaccotto C., Stochastic Modeling of Interest Rates: Actuarial vs. Equilibrium Approach, The Journal of Risk and Insurance.

Cairn A., (2004), Interest Rate Models, Princeton University Press.

Semir C. and Taray M., (2005), Türk Lirası Referans Faiz Oranı-TRLIBOR Uygulaması, Bankacılar Dergisi, Sayı 54.