T.C

(MASTER THESIS)

YAŞAR UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

K-NONSINGULAR ABELIAN GROUPS

Surajo IBRAHIM ISAH

Thesis Supervisor: Prof. Dr. Refail ALIZADE

MATHEMATICS DEPARTMENT

Bornova-IZMIR

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APPROVAL PAGE

This study titled "K-nonsingular Abelian Groups" and presented as Master Thesis by Surajo IBRAHIM ISAH has been evaluated in compliance with the provisions of Yaşar University Graduate Education and Training Regulation and Yaşar University Institute of Science Education and Training Direction. The jury members below have decided for the defense of this thesis, and it has been declared by consensus/majority of the votes that the candidate has succeeded in his thesis defense examination dated................

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ABSTRACT

In this thesis we study *K*-nonsingular modules and in particular *K*-nonsingular abelian groups (Z-modules). Nonsingular (torsion-free) groups are *K*-nonsingular. Direct summands of *K*-nonsingular groups are *K*-nonsingular. We prove that an abelian group *A* is *K*-nonsingular if and only if its torsion part T(*A*) is semisimple and for each prime *p*, $A/\mathrm{T}(A)$ is *p*-divisible if T(*A*) has a direct summand isomorphic to Z_p. In particular a torsion group is *K*-nonsingular iff it is semisimple.

Keywords: *K*-nonsingular modules, *K*-nonsingular abelian groups, torsion groups, torsion-free groups, basic subroups, semisimple modules.

ÖZET

Bu tezde K-tekil olmayan modüller ve özellikle K-tekil olmayan değişmeli gruplar (Z-modüller) incelenmiştir. Tekil olmayan (burulmasız) gruplar K-tekil olmayandır. K-tekil olmayan grupların dik toplam terimleri de K-tekil olmayandır. Bir A değişmeli grubunun K-tekil olmayanlığı için, bunun $T(A)$ burulma alt grubunun yarıbasit olmasının ve bir p asal sayısı için $T(A)$'nın, Z_p'ye izomorf alt grup içermesi durumunda $A/T(A)$'nın p-bölünebilir olmasının gerek ve yeterli olduğunu kanıtladık. özel durumda, K-tekil olmayan burulma grupları tam olarak yarıbasit gruplardır.

Anahtar kelimeler: K-tekil olmayan modüller, K-tekil olmayan değişmeli gruplar, burulma grupları, burulmasız gruplar, temel al gruplar, yarıbasit modüller.

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TEXT OF OATH

I declare and honestly confirm that my study, titled "*K*-nonsingular Abelian Groups" and presented as a Master's Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the references, and that I have benefited from these sources by means of making references.

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Student Signature

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INDEX OF SYMBOLS AND ABBREVIATIONS

Z	The group of integers
Z_n	Integers modulo n
R	Ring
Q	The group of rational numbers
A/B	The quotient group of A mod B
Z_{p^∞}	The primary part of the quotient group Q/Z
nA	The sets of all na with $a \in A$.
dA	Maximal divisible subgroup of an abelian group A
T(A)	Torsion subgroup of an abelian group A
ΣA_i	Sum of abelian groups $A_{i's}$
$\oplus$	Direct sum
ΠA_i	Direct product of groups $A_{i's}$
Soc (A)	Socle of a group A
End (A)	Set of all endomorphisms of A
Ker f	Kernel of a map f
Im f	Image of a map f
$\cong$	Isomorphic
$\leq$	Substructure
$\trianglelefteq$	Essential Substructure

$\leq_{max}$	Maximal Substructure
$\leq_{\oplus}$	Direct summand
$\leq_{pure}$	Pure Substructure
$\forall$	Universal quantifier
$\exists$	Existential quantifier
$\in$ ($\notin$)	Membership (Nonmembership)
$\Rightarrow$ ($\Leftrightarrow$)	Implication (Double implication)
$=$ ($\neq$)	Equals (Not equals)
$\cap$	Intersection

CHAPTER ONE

INTRODUCTION

The property of singularity and nonsingularity of modules in general has variety of applications and has been intensively used in literature. Consider the set $L = \{m \in M : Im = 0 \text{ for some } I \trianglelefteq R\}$, where R is a ring and M is an R-module ($\trianglelefteq$ stand for essential substructure: see chapter two). L is a submodule of M which is called the singular submodule of M. M is called singular module if $L = M$ and M is nonsingular if $L = 0$ (i.e. no nonzero element has essential annihilator in R) [6]. K-nonsingularity is one of the generalized notions of nonsingularity introduced in 2007 by S. Tariq Rizvi and Cosmin S. Roman [7]. A right R-module M is said to be *K-nonsingular* provided that for any $\varphi \in S = End\,(M)$, $r_M(\varphi) = Ker\varphi \trianglelefteq M$ implies that $\varphi = 0$ [7].

The main purpose of our work is to study K-nonsingularity and give characterization of K-nonsingular abelian groups. The work is inspired by some basic theorems of abelian groups [2] and some notions studied in several papers like those in [6] and [7]. In chapter two, we present a review of some of the needed background materials that are helpful for proper understanding of the main work in this thesis. Proofs were sometimes given. For details on more common concepts used, the reader should refer to standard texts more especially on Rings, Modules and Abelian groups (e.g. [2], [5] & [8]).

Chapter three conveys the main work of this thesis. Here we state and prove a characterization of nonsingular abelian groups, we have shown that a torsion-free group is *K*-nonsingular and we present several results through lemmas and propositions that lead us to a characterization of *K*-nonsingular groups. Examples where also provided to give more highlight on these types of group.

The beauty of Mathematical concepts often lies in area of application. One of the areas for which the concept of *K*-nonsingularity is applicable is in type theory. Some of these applications were provided in [7], Rizvi and Roman have provided application of *K*-nonsinguarity to various generalizations of injectivity [7].

CHAPTER TWO

PRELIMINARIES

2.1 Abelian groups

Definition2.1.1. [1, 4.1] A *group* $\langle G, *\rangle$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

(i) For all $a, b, c \in G$, we have

$$a * (b * c) = (a * b) * c. \quad \text{(associativity of } *)$$

(ii) There is an element e in G such that for all $a \in G$,

$$e * a = a * e = a. \quad \text{(identity element } e \text{ for } *)$$

(iii) Corresponding to each $a \in G$ there is an element a' in G such that

$$a * a' = a' * a = e. \quad \text{(inverse } a' \text{ of } a)$$

Definition2.1.2. [3, p.41] A group G is said to be *Abelian* if $a * b = b * a$ for all $a, b \in G$.

‘*’The word abelian derives from the name of the great Norwegian mathematician Niels Henrik Abel (1802-1829), one of the greatest Scientists Norway has ever produced* [3, p.41].’’

Definition2.1.3. [1, 5.4] If a subset B of a group A is closed under the binary operation of A and if B with the induced operation of A is itself a group, then B is a *subgroup* of A.

Notation: $B \leq A$.

For abelian groups, it is habitual to denote the operation additively using the ‘’ + ‘’ sign operation. 0 represent the identity element and the inverse of an element a is denoted by $-a$.

Remark2.1.4. From now on whenever we make mention of the term “group” in short it will mean “abelian group” and we will often represent it by the letter ‘A’

The sum $a + \cdot \cdot \cdot + a$ [n summands] is abbreviated as na, and $- a - \cdot \cdot \cdot - a$ [n summands] as $-na$. $0a = 0 \; \forall \; a \in A$.

The order of a group A is the cardinal number $|A|$ of the set of its element. If $|A|$ is a finite [countable] cardinal, A is called a finite [countable] group.

A subgroup of A always contains the zero of A, and a nonempty subset B of A is a subgroup of A if and only if $a, b \in B$ implies $a + b \in B$ and $a \in B$ implies $-a \in B$, or more simply if and only if $a - b \in B$. The trivial subgroups of A are A and the subgroup consisting of 0 alone. A subgroup of A, different from A, is a proper subgroup of A.

If $B \leq A$ and $a \in A$, the set $a + B = \{a + b \mid b \in B\}$ is called a coset of A modulo B [2, p. 2 & 3].

Definition2.1.5. [2, p.3] The cosets of A mod B form a group A/B known as the quotient or factor group of A mod B.

For $C = a + B$ and $C' = a' + B \in A/B$; $C + C' = (a + a') + B$, $nC = na + B$ and $-C = -a + B$ and the zero of A/B is B.

Definition2.1.6. For an abelian group A and $a \in A$, if all elements $a, 2a, 3a, \ldots, na, \ldots$ are different, we say that the order of a is infinite : $o(a) = +\infty$. If for some $n > m$, we have $na = ma$ then $(n - m)a = 0 \Rightarrow$ there is a minimal $s \in Z^+$ with $sa = 0$, then s is called the order of a: $\mathrm{o}(a) = s$. In this case $a, 2a, 3a, \ldots, (s - 1)\, a, sa = 0$, are different. The set $\{na \mid a \in A, n \in Z\}$ is a subgroup of A, it is called the cyclic subgroup generated by a, and is denoted by $\langle a \rangle$. If $A = \langle a \rangle$ then A is called cyclic group generated by a. [8]

Note that if $o(a) = +\infty$, then $\langle a \rangle = \{0, a, 2a, \ldots, -a, -2a, \ldots\}$. In this case $\langle a \rangle \cong Z$. If $o(a) = n$, then $\langle a \rangle = \{0, a, 2a, \ldots, (s - 1)a\}$. In this case $\langle a \rangle \cong Z_n$. [8]

Definition2.1.7. [2, p.4] If every element of A is of finite order, A is called *torsion group*, while A is *torsion-free* if all its elements, except for 0, are of infinite order. Mixed groups contain both nonzero elements of finite and elements of infinite order. A *primary group* or *p-group* is defined to be a group the orders of whose elements are powers of a fixed prime p.

Theorem2.1.8. [2, 1.1] The set $T(A)$ of all elements of finite order in a group A is a subgroup of A . $T(A)$ is a torsion group and the quotient group $A/T(A)$ is torsion-free.

Remark2.1.9. A is a torsion group $\Leftrightarrow A = T(A)$ and A is torsion-free $\Leftrightarrow T(A) = 0$, i.e. A is torsion-free $\Leftrightarrow$ for any $a \in A$, $o(a) = +\infty$.

Definition2.1.10. [2, p.36-38] Let B, C be subgroups of A, and assume that they satisfy

i) $B + C = A$;

ii) $B \cap C = 0$.

In this case we call A the [internal] direct sum of its subgroups B, C, and write

$$A = B \oplus C.$$

Condition (i) states that every $a \in A$ may be written in the form $a = b + c$ ($b \in B$, $c \in C$), and (ii) amounts to the uniticity of this form.

Let B_i ($i \in I$, : I is an indexing set) be a family of subgroups of A , subject to the following two conditions :

i) $\sum B_i = A$ [i.e. the B_i together generate A] ;

ii) For every $i \in I$, $B_i \cap \sum_{j \neq i} B_j = 0$.

Then A is said to be the direct sum of its subgroups B_i, in sign: $A = \oplus_{i \in I} B_i$.

A subgroup B of A is called a *direct summand* of A, if there is a $C \leq A$ such that $A = B \oplus C$. In this case C is a *complementary direct summand*, or simply C a *complement* of B in A.

One of the most useful properties of direct sums is that: if $A = B \oplus C$, then $C \cong A/B$.

Definition2.1.11. Let A_i be some groups, $i \in I$. The Cartesian product A of $A_{i's}$, $A = \Pi_i A_i$ is a subgroup with operation $(\cdots, a_i, \cdots) + (\cdots, b_i, \cdots) = (\cdots, a_i + b_i, \cdots)$. This group is called the external direct product of groups $A_{i's}$. Elements of A are denoted (a_i), thus $A = \{(a_i) \mid a_i \in A_i\}$

Let $B = \{(a_i) \in \Pi A_i \mid a_i = 0$ for all i except finite number of $i\}$, then $B \leq A$, and it is called the external direct sum of groups A_i's denoted by $B = \oplus A_i$. [8]

Theorem2.1.12. [9, 10.7] (Primary decomposition) Every torsion group A is a direct sum of p-primary groups.

Definition2.1.13. [9, p.309 & 320] If $a \in A$ and n is a nonzero integer, then a is divisible by n in A if there is $b \in A$ with $nb = a$. A group A is divisible if each $a \in A$ is divisible by every nonzero integer n ; that is , there exists $b_n \in A$ with $nb_n = a$ for all $n \neq 0$. (A is divisible implies $nA = A$ for all $n \neq 0$).

Some properties of divisibility include:

(a) If $(n, o(a)) = 1$, then the equation $nb = a$ is always solvable. For if r, s are integers such that $nr + o(a)s = 1$, then $b = ra$ satisfies $nb = nra = nra + o(a)sa = a$.

(b) A group D is divisible if and only if it is p-divisible for every prime p.

If $pD = D$ for every prime p and $n = p_1 \dots p_r$, then $nD = p_1 \dots p_rD = D$.

(c) A p-group is divisible if and only if it is p-divisible.

In view of (b), for a p-group D we always have $qD = D$, whenever the primes p, q are different.

(d) A direct sum or direct product of groups is divisible if and only if each component is divisible.

(e) If D_i $(i \in I)$ are divisible subgroups of A, then so is their sum $\sum D_i$. [2, p.98]

Remarks2.1.14. The quotient group Q/Z is torsion and its p-subgroup, $(Q/Z)_p$, denoted by Z_{p^∞}, is computed as follows:

$\frac{m}{n} + Z \in Z_{p^\infty} \Leftrightarrow p^k(\frac{m}{n} + Z) = Z$ for some k > 0. Thus $p^k(\frac{m}{n}) \in Z$, therefore n must divide p^k. So $n = p^s$ for some $s > 0$. Hence $Z_{p^\infty} = \{\frac{m}{p^s} + Z \mid m \in Z, s \in Z^+\}$. By theorem2.1.12, we have $Q/Z = \oplus Z_{p^\infty}$.

Denote $c_n = \frac{1}{p^n} + Z$. We observe that Z_{p^∞} is generated by the elements $c_1, c_2, c_3, \ldots$ and $pc_1 = 0, pc_2 = c_1, \ldots pc_{n+1} = c_n, \ldots$ Also $o(c_n) = p^n$, hence $\langle c_n\rangle \cong Z_{p^n}$. $\langle c_1\rangle \subseteq \langle c_2\rangle \subseteq \ldots \subseteq \langle c_n\rangle \ldots$

Moreover $Z_{p^n} \cong \langle p\rangle \leq Z_{p^{n+1}}$. [8]

Proposition2.1.15 All subgroups of Z_{p^∞} are 0, Z_{p^∞}, $\langle c_1\rangle, \langle c_2\rangle, \ldots, \langle c_n\rangle, \ldots$ [8]

Corrollary2.1.16. [9, 10.24] If a divisible group D is a subgroup of A, then D is a direct summand of A.

Theorem2.1.17. [9, 10.28] Every divisible group D is a direct sum of copies of Q and of copies of Z_{p^∞} for various p.

Definition2.1.18. [9, p.321 & 322] If A is a group, then dA (i.e. divisible part of A) is the subgroup generated by all the divisible subgroups of A. A group A is *reduced* if $dA = 0$.

Of course, A is divisible if and only if $dA = A$.

Examples2.1.19. i) Every quotient group of a divisible group is divisible.

Let $B \leq A$ where A is divisible. Let $a + B \in A/B$ and $0 \neq n \in Z$. Since A is divisible we have $a = nc$ for some $c \in A$. Then $a + B = nc + B = n(c + B)$. Thus A/B is divisible. □

ii) The quotient group $\Pi Z_p / \oplus Z_p$ is divisible.

Proof:

It suffices to show that $\Pi Z_p / \oplus Z_p$ is divisible by any prime q.

Let $a + \oplus Z_p$ be any element from $\Pi Z_p / \oplus Z_p$, $a = (a_p)$ and q be any prime.

For every $p \neq q$, since gcd $(q, o(a)) = 1$, then $q|a_p$, that is $a_p = qb_p$ for some $b_p \in Z_p$.

Define $c = (c_p) \in \Pi Z_p$ by $c_q = 0$ and $c_p = b_p$ if $p \neq q$. Then $a - qc \in \oplus Z_p$, (for its coordinates are all 0 except for a_q in position q), and $q(c + \oplus Z_p) = qc + \oplus Z_p = a - (a - qc) + \oplus Z_p = a + \oplus Z_p$, so $a + \oplus Z_p$ is divisible by any prime q. Hence $\Pi Z_p / \oplus Z_p$ is divisible. □

Definition2.1.20. [2, p.113] A subgroup B of A is called *pure*, if the equation $na = b$ with $b \in B$, is solvable in B, whenever it is solvable in the whole group A. This amount to saying that B is pure in A if b is divisible by n in A implies b is divisible by n in B.

Remark2.1.21. B is pure in A if and only if $nB = B \cap nA$ for every $n \in Z$.

Examples2.1.22. Every direct summand is pure: In particular a divisible subgroup is pure. If $B \leq A$ and A/B is torsion-free, then B is a pure subgroup of A: In particular torsion part of a group A, T(A) , is pure.

Definition2.1.23. [9, p.326] A subgroup B of a torsion group A is a *basic subgroup* if:

1) B is a direct sum of cyclic groups;
2) B is a pure subgroup of A; and
3) A/B is divisible.

Theorem2.1.24. Every torsion group A has a basic subgroup. (see [9], 10.36)

Corollary2.1.25. [9, 10.41] A pure subgroup S of bounded order, (i.e, $nS = 0$ for some $n > 0$), is a direct summand.

Definition2.1.26. [2, p.136] By p-basic subgroup B of A we mean a subgroup of A satisfying the following three conditions:

(i)B is a direct sum of cyclic p-groups and infinite cyclic groups;

(ii)B is pure in A;

(iii)A/B is p-divisible.

Theorem2.1.27. [2, 32.3] Every group contains p-basic subgroups, for every prime p.

2.2. Module

Definition2.2.1. [1, 18.1] A *ring* $\langle R, +, \cdot \rangle$ is a set R with two binary operations ‘’ + ’’ and ‘’ · ‘’, which we called addition and multiplication, defined on R such that the following axioms are satisfied:

i) $\langle R, + \rangle$ is an abelian group,

ii) Multiplication is associative,

iii) For all $a, b, c \in R$, the left distributive law, $a \cdot (b + c) = a \cdot b + a \cdot c$ and the right distributive law $(a + b)\cdot c = a \cdot c + b \cdot c$ hold.

A subring I of a ring R is called an ideal if for any $r \in R$ and $a \in I$ we have ra and $ar \in I$.

Definition2.2.2. [1, 18.14] A ring in which the multiplication is commutative is a commutative ring. A ring with a multiplicative identity element is a ring with unity, the multiplicative identity element 1 is called unity.

An element u in R with unity $1 \neq 0$, is a unit if it has a multiplicative inverse in R. If every nonzero element in R is a unit then R is a division ring. A commutative division ring is called a field. [1]

Definition2.2.3. Let R be a ring and $(M, +)$ be an abelian group. Suppose that there is a function $f : R \times M \rightarrow M$ (we will denote $f(r, m)$ by rm, where $r \in R$ and $m \in M$) such that the following conditions are satisfied:

1) $r(m + n) = rm + rn$ for every $r \in R$ and $m, n \in M$

2) $(r + s)m = rm + sm$ for every $r, s \in R$ and $m \in M$

3) $(rs)m = r(sm)$ for every $r, s \in R$ and $m \in M$.

Then we say that M is a left R-*module*, (or simply a *module*).

If $f : M \times R \to M$ exists with similar conditions, M is a right R-module.

Usually R is a ring with unity 1 and $1.m = m$ for every $m \in M$ [10].

A subset N of an R-module M is called a submodule of M if N is itself a module with respect to the same operations. Notation: same as for subgroup.

A submodule N of M is called a maximal submodule of M if $N \leq K \leq M$ implies $K = N$ or $K = M$.

It is clear that a module is just like a vector space over a ring R and an abelian group is a Z-module.

2.3 Semisimple module

Definition2.3.1. A module S is a simple if it has no proper nonzero submodule; i.e. S has only 0 and itself as submodules. Equivalently $0 \leq X \leq S \Rightarrow X = 0$ or $X = S$. [10]

Remark2.3.2. It is not difficult to see that a simple abelian group is precisely Z_p upto isomorphism, for some prime number p. Thus a simple abelian group must be a finite cyclic group of prime order.

Theorem2.3.3. [5, 8.1.3] For a module M the following are equivalent:

1) Every submodule of M is a sum of simple submodules.
2) M is a sum of simple submodules.

3) M is a direct sum of simple submodules.

4) Every submodule of M is a direct summand of M.

If any of the conditions of theorem2.3.3 above is satisfied, then the module M is called a semisimple [10].

Examples2.3.4. Every vector space V_K over a field K is semisimple. An abelian group A is semisimple if and only if $A \cong \oplus Z_p$. **Q** and Z are not semisimple since they have no simple subgroups. Every sum of semisimple module is semisimple and submodules of semisimple modules are semisimples.

2.4 Essential submodule

The definitions and theorems given in this section and section2.5 can be found in [5].

Definition2.4.1. A submodule N of M is *essential* (big or large) in M if $N \cap K = 0$ for some $K \leq M$ implies $K = 0$.

Notation: $N \trianglelefteq M$.

Remark2.4.2. It is clear from the defition that

1) $N \trianglelefteq M$ iff $\forall\ 0 \neq K \leq M$, $N \cap K \neq 0$

2) If $M \neq 0$ and $N \trianglelefteq M$ then $N \neq 0$

3) $M \trianglelefteq M$.

Lemma2.4.3. $N \trianglelefteq M$ if and only if for every $0 \neq m \in M$ there is $r \in R$ such that $0 \neq rm \in N$.

Proof:

Let $N \trianglelefteq M$ and $0 \neq m \in M$. Then $0 \neq Rm \leq M$. Therefore, $N \cap Rm \neq 0$, hence $0 \neq rm \in N$, for some $r \in R$.

Conversely, let $0 \neq K \leq M$, then $\exists\ 0 \neq k \in K \leq M$. By hypothesis we have $0 \neq rk \in N \cap K$, for some $r \in R$, so $N \cap K \neq 0$, therefore $N \trianglelefteq M$. □

Definition2.4.4. The *Socle* of a module *M* is the intersection of all essential submodules of *M*, equivalently Socle is the sum of all simple submodules of *M*.

At this junction we will state the isomorphism theorems which we shall often use in the next chapter.

2.5. Isomorphism Theorems

Definition2.5.1. A function $f : M \to N$ is a homomorphism if $f(a + b) = f(a) + f(b)$ and $f(ra) = rf(a)$, where *M, N* are modules over *R* , $a, b \in M$ and $r \in$ R.

Definition2.5.2. An endomorphism is a homomorphism of *M* into *M*.

For brevity, the set of all endomorphisms of *M* is denoted by *End (M).*

Definition2.5.3. The *kernel* of a homomorphism *f* defined on *M* is the set of all elements in *M* that are mapped to zero, i.e. $Ker\ f = \{ m \in M \mid f(m) = 0 \}$.The *Image* of *f*, $\text{Im}\ f = \{ f(m) \mid m \in M \}$.

The kernel and image of *f* are submodules of *M* and *N* respectively.

Definition2.5.4. An onto homomorphism is called an epimorphism (epic); one to one homomorphism is called a monomorphism (monic); one to one and onto (bijective) homormophism is called an isomorphism, in this case M and N are said to be isomorphic. Notation: $M \cong N$.

Remark2.5.5. f is monic if and only if $Ker\, f = 0$ and f is epic if and only if $Im\, f = N$.

Examples of homomorphisms includes: The natural (canonical) homomorphism , σ , of a module A onto the factor module A/B , where $B \leq A$; $\sigma : A \to A/B$ defined by $\sigma(a) = a + B$. The identity injection or inclusion map of submodule $B \leq A$; $i : B \to A$ defined by $i(b) = b$, and the natural projection map $\pi : \Pi A_i \to A_j$ defined by $\pi(a_i) = a_j$.

Theorem2.5.6. *Every module homomorphism $f : M \to N$ has a factorization $f = g \circ \sigma$, where $\sigma : M \to M/Kerf$ is the canonical epimorphism and $g : M/Kerf \to N$ is defined by $g(m + Kerf) = f(m)$. Moreover g is an isomorphism iff f is an epimorphism.*

Theorem2.5.7. *Fundamental Homomorphism Theorem*

For every homomorphism $f : M \to N$, $M/Ker\, f \cong Im\, f$

In particular if f is an epic then $M/Ker\, f \cong N$.

<u>*Proof:*</u>

$f' : A \to Imf$ defined by $f'(m) = f(m)$ is an epimorphism. Therefore $g : M/Kerf \to Imf$ is an isomorphism by theorem2.5.6. □

Theorem2.5.8. ***Second Isomorphism Theorem***

If N and K are submodules of M, then

$$(N + K)/K \cong N/(N \cap K)$$

Proof:

Define $f : N \to (N + K)/K$ by $f(n) = n + K$. Then f is a homomorphism.

For every $(n + k) + K \in (N + K)/K$, we have $(n + k) + K = n + K = f(n) \Rightarrow f$ is epic.

$n \in Kerf \Leftrightarrow f(n) = 0 \Leftrightarrow n + K = K \Leftrightarrow n \in K \Leftrightarrow n \in N \cap K \Leftrightarrow Kerf = N \cap K$.

f is epic. $\Rightarrow Imf = (N + K)/K \cong N/Kerf = N/N \cap K$ (by theorem2.5.8). □

Theorem2.5.9. ***Third Isomorphism Theorem***

If $K \leq N \leq M$, then

$$(M/K) / (N/K) \cong M/N$$

Proof:

Define $f : M/K \to M/N$ by $f(m + K) = m + N$. f is well defined because for $m_1 + K = m_2 + K$ we have $m_1 - m_2 \in K \leq N \Rightarrow m_1 + N = m_2 + N$.

It is clear that f is epic.

$m + K \in Kerf \Leftrightarrow f(m + K) = 0 \Leftrightarrow m + N = N \Leftrightarrow m \in N \Leftrightarrow Kerf = N/K$.

By theorem2.5.8 we have $M/N = Imf \cong (M/K)/Kerf = (M/K)/(N/K)$. □

We conclude this section with definition and a fundamental result on *K*-nonsingular modules that we have used in the next chapter.

2.6. K-nonsingular modules

Definition2.6.1.(Rizvi and Roman, 2007). Let *M* be a module. The singular submodule of *M* is defined by

$$Z(M) = \{m \in M \mid Im = 0 \text{ for some } I \trianglelefteq R\}.$$

If $Z(M) = M$, then *M* is called singular module, dually *M* is nonsingular provided $Z(M) = 0$.

Definition2.6.2 (Rizvi and Roman, 2007) A module *M* is called *K*-nonsingular if, for every $\varphi \in End(M)$, $Ker\varphi \trianglelefteq M$ implies $\varphi = 0$.

Example2.6.3. Any semisimple module is *K*-nonsingular, this follows from definition2.6.3 and Theorem2.3.2 [7].

Proposition2.6.4.(Rizvi and Roman, 2004). *If M is a nonsingular module then M is K-nonsingular*.

Proof:

Suppose to the contrary that *M* is not *K*-nonsingular, then $\exists\ 0 \neq \varphi \in S$ such that $Ker\varphi \trianglelefteq M$. Since $\varphi \neq 0$, $\exists\ 0 \neq m \in M \backslash Ker\ \varphi$. The set $I = \{r \in R : mr \in Ker\varphi\}$ is a right ideal in *R*. In fact, $I \trianglelefteq R : r \notin I \Rightarrow mr \notin Ker\varphi \Rightarrow \exists\, r'$ such that $0 \neq mrr' \in Ker\varphi \Rightarrow 0 \neq rr' \in I$. But for $0 \neq \varphi(m)$, $\varphi(m)I = 0$, contradiction with the nonsingularity of *M*. □

Definition2.6.5 (Rizvi and Roman, 2007) A module *M* is polyform if and only if for any $K \subseteq M$ and $0 \neq f : K \to M$, *Kerf* is not essential in *K*.

CHAPTER THREE

In this chapter, we are going to focus on *K*-nonsingular Abelian groups (*Z*-modules). Some examples and important lemmas and propositions concerning the *K*-nonsingular Abelian groups will be discussed. Most of these collectively lead us to a characterization of the *K*-nonsingular Abelian groups.

3.1 *K-NONSINGULAR ABELIAN GROUPS*

Recall that an abelian group *A* (a *Z*-module) is called *K*-nonsingular if, for every $\varphi \in End(A)$, $Ker\varphi$ is essential subgroup of *A* implies that $\varphi = 0$.

In other words, an abelian group *A* is *K*-nonsingular if for every nonzero endomorphism of *A*, its kernel in not an essential subgroup of *A*.

Examples3.2

1. The group Z_{p^∞} is not a *K*-nonsingular group: all its nonzero subgroups are essential subgroups. Also the group Z_4 is not *K*-nonsingular for $\varphi : Z_4 \to Z_4$ defined by $\varphi(a) = 2a$, we have $Ker\varphi = \{0,2\}$ which is an essential subgroup of Z_4.

2. As we have seen in the previous chapter semisimple abelian groups (semisiple Z-modules) are *K*-nonsingular. Simple abelian groups are exactly cyclic groups of a prime order, so direct sums of groups isomorphic to Z_p for some primes p are *K*-nonsingular.

i. $\oplus Z_p$ and Z_p where p is a prime are *K*-nonsingular groups.

ii. Any cyclic group of prime order is *K*-nonsingular

3. Z_n is K-nonsingular where n is square-free: this follows from the fact that Z_n is semisimple if and only if n is square-free.

We recall that a group A is called *polyform* if and only if for any $B \leq A$ and $f: B \rightarrow A$, $Kerf$ is not essential in B.

Corollary3.3. *Any polyform group A is K-nonsingular*.

Proof:

Let A be a polyform group, $B \leq A$ and $f: B \rightarrow A$. Then $Kerf$ is not essential in B In particular for $B = A$, all nonzero endomorphism of A have Kernels which are not essential in A, hence the assertion is proved. □

Next, we give a characterization of nonsingular abelian groups. From the definition of singular submodules, it is clear that for groups, i.e. Z-modules, the notion coincide with that of the torsion part. This is due to the fact that every ideal of Z is essential in Z. We therefore have the following lemma.

Lemma3.4. *A group A is nonsingular if and only if A is torsion-free*.

Proof:

A group A is nonsingular if and only if $Z(A) = 0 \Leftrightarrow \forall a \in A$ such that $ak = 0$, for some $k \in nZ$, implies $a = 0 \Leftrightarrow A$ is torsion-free. □

Proposition3.5. If A is a nonsingular group, then A is K-nonsingular.

The converse does not hold generally, showing that the property of nonsingularity is stronger than the K- nonsingularity.

Proof:

See the proof in chapter two given in general module theoretic setting. □

Corollary3.6. *Every torsion free group is K- nonsingular*.

Proof:

A is torsion-free $\Leftrightarrow$ A is nonsingular $\Rightarrow$ A is K-nonsingular, by lemma3.4 and proposition 3.5 respectively. □

To show that the converse of the above corollary is indeed not necessarily true we consider the following counter example.

Example 3.7. Z_n where n is prime is K-nonsingular because it is simple. But it is not nonsingular since for any $x \in Z_n$ we have $x.nZ = 0$, and $nZ \trianglelefteq Z$.

Now, we shall look at some important lemmas and immediate consequent results obtained as follows. Before that we have a corollary:

Corollary3.8

(a) Any cyclic group of infinite order is K-nonsingular.

(b) For any group A, the quotient group $A/\mathrm{T}(A)$ *is K-nonsingular, where* $\mathrm{T}(A)$ *is the torsion subgroup of A.*

Proof:

It follows from the facts that infinite cyclic group is isomorphic to Z and that $A/T(A)$ is torsion-free. □

Lemma3.9. *If $C \trianglelefteq A$ then $C \oplus B \trianglelefteq A \oplus B$ for every module B.*

Proof:

Assume that $C \trianglelefteq A$ and let $0 \neq a + b \in A \oplus B$.

We need to show that there exists $r \in R$ such that $0 \neq r(a + b) \in C \oplus B$.

If $a = 0$, then $1(a + b) = b \in C \oplus B$.

If $a \neq 0$, then $0 \neq ra \in C$, for some $r \in R$, since $C \trianglelefteq A$. Therefore $0 \neq ra + rb \in C \oplus B$, as $A \cap B = 0$ and $0 \neq ra$, thus $0 \neq r(a + b) \in C \oplus B$, hence our result. □

Lemma3.10. *A direct summand of a K-nonsingular module is K-nonsingular.*

Proof:

Let A be a K-nonsingular module and B be a direct summand of A such that $A= B \oplus C$ for some $C \leq A$.

Let $\varphi \in End(B)$ such that $Ker\varphi \trianglelefteq B$.

Define $\psi : A \to A$ by $\psi = i_B \circ \varphi \circ \pi_B$, where i_B and π_B is the inclusion map and the canonical projection on B respectively. Then $\psi \in End(A)$ and $Ker\psi = Ker\varphi \oplus C \trianglelefteq B \oplus C = A$ (by lemma3.9). Therefore $Ker\ \psi \trianglelefteq A$ and since A is K-nonsingular we must have $\psi = 0 \Rightarrow \varphi = 0$ (as neither i_B nor π_B is zero). Thus $Ker\varphi \trianglelefteq B$ implies $\varphi = 0$, hence B is K-nonsingular. □

Lemma3.11. *Let p be a prime integer. The group Z_{p^n} is K-nonsingular if and only if $n = 1$.*

Proof:

(⇒) Suppose that $n \neq 1$, i.e. $n \geq 2$.

Define $\varphi : Z_{p^n} \rightarrow Z_{p^n}$ by $\varphi(x) = px$, then φ is a nonzero endomorphism of Z_{p^n} .

$Ker\varphi = \{x \in Z_{p^n} \mid \varphi(x) = px = 0\} \cong \langle c_{n-1} \rangle \cong Z_{p^{n-1}} \cong \langle p \rangle \trianglelefteq Z_{p^n}$, hence $Ker\varphi \trianglelefteq Z_{p^n}$.

So, $Ker\varphi \trianglelefteq Z_{p^n}$ with $\varphi \neq 0$ therefore Z_{p^n} is not K-nonsingular.

(⇐) The converse is trivial because for $n = 1$, Z_{p^n} is simple which is K-nonsingular.□

Proposition3.12. *If an abelian group A is K-nonsingular then its torsion part T(A) is semisimple.*

Proof:

Let A be a K-nonsingular group, $T(A)$ be the torsion part and $T_p(A)$ be its p-component.

If $d(T_p(A)) \neq 0$ then $A \cong Z_{p^\infty} \oplus X$, therefore Z_{p^∞} must be K-nonsingular, contradiction.

So $T_p(A)$ is reduced.

Let $B_p(A)$ be its basic subgroup. $B_p(A) = \oplus_{i\in I} \langle b_i \rangle$ where $o(b_i) = p^{n_i}$. For each $i \in I$ we have $\langle b_i \rangle \leq_{\oplus} B_p(A) \leq_{pure} T_p(A) \leq_{\oplus} T(A) \leq_{pure} A$.

Therefore $\langle b_i \rangle$ is a direct summand in A and so is K-nonsingular by lemma3.10. Using lemma3.11 we get that $n_i = 1$ for every $i \in I$. So $B_p(A)$ is semisimple.

Moreover $B_p(A) \leq_{pure} T_p(A)$ and $B_p(A)$ is bounded, therefore $T_p(A) = B_p(A) \oplus D$, where D is divisible. But $T_p(A)$ is reduced, hence $D = 0$, i.e. $T_p(A) = B_p(A)$ is semisimple.

Thus $T(A) = \oplus T_p(A)$ is semisimple. □

Example3.13. (a) The group $\Pi_{p\in P} Z_p$ is K-nonsingular.

For any endomorphism $f : \Pi Z_p \to \Pi Z_p$ with $Kerf \trianglelefteq \Pi Z_p$, $\oplus Z_p = Soc(\Pi Z_p) \subseteq Kerf$.

So $Imf \cong (\Pi Z_p / Kerf) \cong (\Pi Z_p / \oplus Z_p)/(Kerf / \oplus Z_p)$ is divisible since the group $\Pi Z_p / \oplus Z_p$ is divisible. But ΠZ_p is reduced, hence $Im f = 0$, that is $f = 0$.

Note that: $T(\Pi Z_p) = \oplus Z_p$ is semisimple.

(b) Torsion-free groups are K-nonsingular; their torsion part is 0 which is semisimple.

Lemma3.14. *A maximal submodule B of a module A is either essential or a direct summand in A*

Proof:

Let $B \leq_{max} A$. Suppose that B is not essential, then there is $0 \neq C \leq A$ such that $B \cap C = 0$.

Then $B \subsetneqq B \oplus C \leq A$. By maximality of B we deduce that $B \oplus C = A$. So B is a direct summand in A. □

Lemma3.15. *If $A \trianglelefteq B$, then B/A is torsion.*

Proof:

Let $0 \neq b \in B$. $\langle b \rangle \cap A \neq 0 \Rightarrow nb \in A$ for some $n \neq 0$.

Then for $b + A \in B/A$, $n(b + A) = nb + A = A$, $n \neq 0$, hence our result. □

Next, we came up with the following main result that gives a characterization of *K*-nonsingular groups.

Theorem3.16. *An abelian group A is K-nonsingular iff $T(A)$ is semisimple and for each prime p, $A/T(A)$ is p- divisible if $T(A)$ has a direct summand isomorphic to Z_p.*

Proof:

($\Rightarrow$) A is K-nonsingular implies that $T(A)$ is semisimple by proposition3.12.

Suppose that $T(A) \cong Z_p \oplus K$ and $A/T(A)$ is not p-divisible.

Define $f : A \to A$ by $f = i \circ \pi \circ \sigma_2 \circ \sigma_1$

Where $\sigma_1 : A \to A/T(A)$, $\sigma_2 : A/T(A) \to (A/T(A))/p(A/T(A)) \cong \oplus Z_p$ are canonical epimorphisms, $\pi : \oplus Z_p \to Z_p$ is the projection on $\oplus Z_p$ and $i : Z_p \to A$ is the inclusion map.

$Z_p \cong Im\, f \cong A/Kerf$, therefore $Kerf$ is a maximal subgroup of A. If $Kerf$ is a direct summand in A, then $A = Kerf \oplus C$ where $C \cong Z_p$, hence $C \leq T(A) \subseteq Kerf$. But then $C = C \cap Kerf = 0 \Rightarrow$ contradiction, so $Kerf$ is not a direct summand in A and by lemma3.14 we have $Kerf \trianglelefteq A$. But f is nonzero, which is a contradiction with the fact that A is K-nonsingular. So $A/T(A)$ is p-divisible.

($\Leftarrow$) Suppose that A is not K-nonsingular, i.e. there is a nonzero endomorphism

$f : A \to A$ with $Ker\, f \trianglelefteq A$.

Then $Imf \cong A/Kerf$ is torsion, therefore $Imf \leq T(A)$.

So Imf is a nonzero semisimple group. Then $Imf \cong Z_p \oplus N$ for some $N \leq Imf$, therefore $A/T(A)$ is p-divisible. Since $Soc(A)$ is the intersection of essential subgroups of A (equivalently sum of all simple subgroups of A), $T(A) = Soc(A) \leq Kerf$.

$Imf \cong A/Kerf \cong (A/T(A))/(Kerf/T(A))$. Then Imf must be p-divisible hence Z_p must be p-divisible, contradiction. So A is K-nonsingular. □

Corollary3.17. *A torsion group A is K-nonsingular iff it is semisimple.*

CHAPTER FOUR

4.1 SUMMARY

Chapter one is the introductory chapter. It gives a brief Historical background of our research.

In chapter two we discussed the basic notions needed for a novice to read and get the concepts without worries. This chapter was concluded with brief study of the *K*-nonsingular modules.

Chapter three carries the main work on *K*-nonsingular groups; in this chapter we have shown that, every torsion-free group is *K*-nonsingular , direct summand of *K*-nonsingular is also *K*-nonsingular, the torsion subgroup, *T(A)*, of *K*-nonsingular group *A* is Semisimple and most importantly we came up with a characterization of the *K*-nonsingular groups (Theorem3.16) after proving several lemmas.

4.2 CONCLUSION

An abelian group *A* is *K*-nonsingular iff T(*A*) is semisimple and for each prime *p*, $A/\mathrm{T}(A)$ is *p*-divisible if T(*A*) has a direct summand isomorphic to Z_p. In particular, a torsion group *A* is *K*-nonsingular iff it is semisimple.

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