WEAK SOLUTIONS OF CAUCHY DYNAMIC AND HYPERBOLIC PARTIAL DYNAMIC EQUATIONS IN BANACH SPACES

A DISSERTATION SUBMITTED TO

THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF YASAR UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

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June 06, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of master of science.

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ABSTRACT

WEAK SOLUTIONS OF CAUCHY DYNAMIC AND HYPERBOLIC PARTIAL DYNAMIC EQUATIONS IN BANACH SPACES

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MS. in Mathematics
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June 06, 2012

In this dissertation, we obtain the sufficient conditions (as close as necessary conditions) for the existence of weak solutions for the second order dynamic Cauchy problem with mixed derivatives

$$\begin{array}{l} x^{\Delta\nabla}(t) = f(t, x(t)), \\ x(0) = 0, \quad x^{\Delta}(0) = \eta_1 \end{array}, \quad t \in \mathbb{T}, \quad \eta_1 \in E$$

and an hyperbolic partial dynamic equation

$$\begin{array}{l} z^{\Gamma\Delta}(x, y) = f(x, y, z(x, y)), \\ z(x, 0) = 0, \quad z(0, y) = 0 \end{array}, \quad x \in \mathbb{T}_1, \quad y \in \mathbb{T}_2$$

in Banach spaces. As the dynamic equations are the unification of differential, difference and q-discrete equations, our results are also true for the special cases $\mathbb{R}, \mathbb{Z}, h\mathbb{Z}$ and $\mathbb{K}_q$.

We establish integral operators corresponding to our problems in appropriate circumstances and we prove the existence of the fixed points of these operators via Sadowskii fixed point theorem. The measure of weak noncompactness β, introduced by DeBlasi,

$$\beta(A) = \inf\{t > 0 : \text{ there exists } C \in K^w \text{ such that } A \subset C + tB_1\},$$

is used for the compactness condition of the operators.

Keywords: Time Scale, fixed point theorem, weak solution, Cauchy problem, measure of weak noncompactness, mean value theorem.

ÖZET

BANACH UZAYLAR ÜZERİNDE CAUCHY DİNAMİK VE KISMİ TÜREVLİ HİPERBOLİK DİNAMİK DENKLEMİNİN ZAYIF ÇÖZÜMLERİ

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Bu tezde, Banach uzaylar üzerinde $\Delta\nabla$-zayıf türevler içeren ikinci mertebeden Cauchy probleminin

$$\begin{array}{l} x^{\Delta\nabla}(t) = f(t, x(t)), \\ x(0) = 0, \quad x^{\Delta}(0) = \eta_1 \end{array}, \quad t \in \mathbb{T}, \quad \eta_1 \in E$$

ve hiperbolik kısmi zayıf türevli dinamik denklemin

$$\begin{array}{l} z^{\Gamma\Delta}(x, y) = f(x, y, z(x, y)), \\ z(x, 0) = 0, \quad z(0, y) = 0 \end{array}, \quad x \in \mathbb{T}_1, \quad y \in \mathbb{T}_2.$$

zayıf çözümlerinin varlığı için yeter koşullar (gerek koşullara mümkün olduğu kadar yakın) elde ettik. Dinamik denklemler, diferansiyel, fark ve q-diskret denklemlerin bir birleşimi olduğundan, sonuçlarımız $\mathbb{R}, \mathbb{Z}, h\mathbb{Z}$ ve $\mathbb{K}_q$ özel durumları olan için de doğrudur.

Öncelikle problemlerimize uygun koşullarda karşılık gelen integral operatörleri oluşturduk ve Sadowskii sabit nokta teoremi yardımı ile bu operatörlerin sabit noktalarının varlığını ispatladık. Operatörlerin kompaktlık koşulları için DeBlasi tarafından geliştirilen β zayıf kompakt olmama ölçümü

$$\beta(A) = \inf\{t > 0 : \exists C \in K^w \text{ öyle ki } A \subset C + tB_1\}$$

kullanıldı.

Anahtar sözcükler: Zaman skalası, sabit nokta teoremi, zayıf çözüm, Cauchy problemi, zayıf kompakt olmama ölçümü,ortalama değer teoremi.

Acknowledgement

First of all, this thesis was composed of great effort in long days. During these days, it is pleasure to thank here Asst. Prof. Ahmet YANTIR, who has been my advisor, for the effort that he has devoted to me, for his insightful remarks which have helped me, and most thanks for his friendly familiarity. I could not finish my thesis without him. I will always estimate his guidance, dedication, and exemplary academic standards.

Dear Prof. Dr. Mehmet TERZİLER, thank for his fervour for me. I will be grateful for his opportunities that he has provided me all the time.

Next, I would like to thank to Asst. Prof. R.Serkan ALBAYRAK for his attention during my entry to university.

And also my-ex teacher Asst. Prof. Burcu Silindir YANTIR. I sincerely thank and owe a great deal to her for her struggle and relevance.

Another special mention is for another my ex-teacher. I would like to thank to Asst. Prof. Uğur MADRAN for providing me the LaTeX knowledge and his experience during this period. He will be more than a teacher for me forever.

Last but not at least I thank to my mother. Without her mutual and natural affection, patience, and opportunities this PhD degree cannot result.

Duygu SOYOĞLU

to my precious mother ...

Contents

Notation

Symbol	Response
$\mathbb{T}$	**Time scale**
$[a,b]_{\mathbb{T}}$	$[a,b]\cap\mathbb{T}$
T_i	$[t_i,t_{i+1}]\cap\mathbb{T}$
P	**The partition of** $[a,b]$
$\mathbb{T}^{\kappa}$	**The region of** $\Delta-$**differentiability**
$\mathbb{T}_{\kappa}$	**The region of** $\nabla-$**differentiability**
$\mathbb{R}$	**The set of real numbers**
$\mathbb{R}^n$	**The Cartesian product of n copies of** $\mathbb{R}$
Ω	**The bounded subset of** $\mathbb{R}^n$
$\mathbb{Z}$	**The set of integers**
S_∞	**The set of all nonnegative real sequences**
C_{rd}	**The set of rd-continuous functions**
E	**Banach space**
K^w	**The set of weakly compact subsets of** E
$h\mathbb{Z}$	$\{hn: n\in\mathbb{Z}, h>0\}$
$\mathbb{N}$	**The set of natural numbers**
$\mathbb{K}_q$	$\{q^n: q\in\mathbf{Q}, q>1, n\in\mathbb{Z}\}\cup\{0\}$
$B(s_k,\varepsilon)$	**The open ball with center** s_k **and radius** ε
B_1	**The unit ball**
σ	**Forward jump operator**
ρ	**Backward jump operator**
μ	**Forward graininess function**
ν	**Forward graininess function**
$\max\mathbb{T}$	**Maximum element of the time scale** $\mathbb{T}$
$\min\mathbb{T}$	**Maximum element of the time scale** $\mathbb{T}$

Symbol	Response
N_t	**A neighborhood of** t
$C([a,b])$	**The set of continuous functions defined on** $[a,b]$
$L^1([a,b])$	**The set of Lebesgue integrable functions defined on** $[a,b]$
$(C(\mathbb{T},E);w)$	**Weakly continuous functions from** $\mathbb{T}$ **to** E
V_j	**Finite or countable cover**
$\mathfrak{F}_1$	**The set of time scale intervals of the form** $[a,b)$
$\mathfrak{F}_2$	**The set of time scale intervals of the form** $[a,b)$
m_1^*	**Outer measure**
$\mathfrak{M}(m_1^*)$	**The set of** m_1^* **measurable sets**
μ_Δ	**Lebesgue** $\Delta-$ **measure**
μ_∇	**Lebesgue** $\Delta-$ **measure**
$conv(A)$	**Convex hull of** A
$mes(I)$	**The measure of the interval** I
$\bar{A}$	**The closure of** A
$\overline{A}^w$	**The weak closure of** A
$\beta(A)$	**The measure of weak noncompactness of** A
$diam(A)$	**The diameter of** A
$\|\|u\|\|$	**The norm of** u
HCP	**Hyperbolic Cauchy problem**
$pp.$	**pages**
$a.e.$	**almost everywhere**

Chapter 1

Introduction

Time scale (or a measure chain) which unifies the discrete and continuous analysis was initiaded by Hilger in his Ph.D. thesis [43]. Hilger formed the definitions of derivatives and integral on time scales. After the theory of time scale was created and the first article by Aulbach and Hilger [14] was published, the concept of the time scale attracted the attention of the scientists especially working on discrete and continuous models. The differential equations, difference equations and quantum equations [45] h-difference (uniform stepsize) and q-difference (nonuniform stepsize) equations were unified as "dynamic equations".

Time scale provides two important advantages for the theory of differential equations. By the "unification" property of time scales differential equations, difference equations with uniform step-size (h-difference equations on $h\mathbb{R}$) and the quantum equations (q-difference (differential) equations on $\mathbb{K}_q$) can be unified. On the other hand, by the extension property the new approach for the theory of differential equations (the differential equations formed on a set which is combination of continuous intervals and discrete sets) can be constructed. The landmark articles for the dynamic equations on time scales are [29, 31, 30, 52, 53].

The first books on time scales on written by Kaymakçalan, Lakshmikantham and Sivasundaram [46], and Bohner and Peterson [18]. The book [20] edited by Bohner and Peterson collected the pinoneering articles on time scales.

Cauchy differential equations $x'(t) = f(t, x(t))$ and Cauchy difference equation have been widely studied by many authors [7, 25, 26, 27, 28, 33, 34, 37, 40, 47, 49, 50, 51, 55, 60, 61] in the literature. Authors studied the existence and properties of classical, weak, Carathéodory and pseudo-solutions of Cauchy problem. The existence of solutions is proved by the fixed point of Mönch [56] or Kubiaczyk [50]. Among them one of the most detailed article is by Cichon [28]. Cichon studied the Cauchy problem

$$\begin{aligned} x'(t) &= f(t, x(t)) \\ x(0) = 0, \quad t &\in I = [0, \alpha] \end{aligned}$$

in details with different type of integrals. In this work author represented the requirements on the nonlinear term f as possible as close to the necessary conditions.

To obtain the existence results and investigate the structure of the solution, he defines the notation of solution in the following forms:

- Classical solution (with continuous f)
- Carathéodory solution (with f differentiable a.e)[24]
- Weak solution (using weak topology on E)[61]
- Pseudo- solution (Carathéodory case with weak topology) [54].

However the dynamic equations in Banach spaces are quite new research area. For Cauchy dynamic equation

$$x^{\Delta}(t) = f(t, x(t))$$

in Banach spaces, the landmark articles belongs to Cichon *et.al* [29, 30]. Authors formed the time scale calculus (and the weak time scale calculus) for Banach space valued functions. By means of measure of noncompactness [16] and Mönch fixed point theorem [56], the sufficient conditions for the existence of classical and Carathéodory type solutions are stated.

The existence of weak solutions, by means of measure of weak noncompactness [36] and the fixed point theorem of Kubiaczyk [50], are obtained [29]. For this purpose the weak Δ-derivative, and the Δ-integral in Banach spaces and the

Mean Value Theorem for weak Δ-integrals are introduced.
Satco [59] obtained the existence of continuous solution for a nonlocal Cauchy problem with integral boundary conditions in Banach spaces, considering non-absolutely convergent Δ-integrals.

This thesis is organized as follows:

In Chapter 2, we first give basic preliminary introduction to time scale concept and time scale calculus. Then we state the deformed definitions of time scale calculus for Banach space valued functions, *i.e.*, for the functions $f : \mathbb{T} \to E$. Next we improve the weak time scale calculus which is introduced by Cichon *et.al.* [29] for double integrals. Consequently we list the theorems and definitions which will be used in the rest of the dissertation. Finally we give brief information about measure of noncompactness β.

In Chapter 3 we prove the existence of weak solutions of the second order dynamic Cauchy problem with mixed derivatives. For this purpose, we make use of measure of DeBlasi weak noncompactness β, the mean value theorem for weak Δ-integrals and Sadowskii and Kubiaczyk fixed point theorems.

In Chapter 4 we generalize the result of [35] and prove the existence of weak solutions of an hyperbolic partial dynamic equation in Banach spaces. The weak Riemann double integral and the mean value theorem for double weak integrals are introduced in Chapter 2 to prove the main result. We make use of DeBlasi measure of weak noncompactness and Sadowskii and Kubiaczyk fixed point theorems.

Chapter 2

Preliminaries

2.1 Time Scale Calculus

A time scale (or a measure chain) is an arbitrary nonempty closed subset of real numbers. Therefore, the set of real numbers, the integers, the natural numbers, and the Cantor set are the examples of time scales. However, the rational numbers, the complex numbers, and the open interval $(0,1)$, are not time scales. The calculus of time scales was iniated by Stefan Hilger in order to create a theory that can unify discrete and continuous analysis [44]. Indeed for the most famous time scales, the delta derivative f^{Δ} for a function f defined on $\mathbb{T}$ turns out to be

1. $f^{\Delta} = f'$ if $\mathbb{T} = \mathbb{R}$,
2. $f^{\Delta} = \Delta f$ if $\mathbb{T} = \mathbb{Z}$,
3. $f^{\Delta} = \Delta\ qf = \dfrac{f(qt) - f(t)}{(q-1)t}$ if $\mathbb{T} = \mathbb{K}_q$.

After Hilger created the theory, many authors contributed the theory of time scales. (see [3, 5, 8, 9, 11, 13, 14, 18, 20, 46] and references therein). For the basic calculus on time scales, we follow the definitions and notations of the books by Bohner and Peterson [18, 20] in the next subsections.
In this dissertation, by the interval $[a,b]_{\mathbb{T}}$, we mean the intersection of the interval

$[a, b]$ and time scale $\mathbb{T}$, i.e., $[a, b]_{\mathbb{T}} = \{s \in \mathbb{T} : a \leq s \leq b\}$. Other types of intervals can be defined in a similar manner.

2.1.1 Forward and Backward Jump Operators

For an easy understanding of the issue of time scale calculus, first we have to define the forward jump operator σ, the backward jump operator ρ, the graininess functions μ and ν, the region of Δ- differentiability $\mathbb{T}^{\kappa}$ and the region of ∇-differentiability $\mathbb{T}_{\kappa}$. The closest point in the time scale on the right and left of a given point are the forward jump and backward jump operators, respectively. And the graininess functions are the distance from a point to the closest point on the right and the left.

Definition 2.1.1 *Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

For the points $M = \max(\mathbb{T})$ and $m = \min(\mathbb{T})$, $\sigma(M) = M$ and $\rho(m) = m$. Clearly, for $t \in \mathbb{T}$, the images $\sigma(t), \rho(t) \in \mathbb{T}$, as $\mathbb{T}$ is closed. The point $t \in \mathbb{T}$ can be classified with respect to the images under the operators σ and ρ as follows:

$$\begin{aligned} t \textit{ is right dense if } \sigma(t) &= t, \\ t \textit{ is right scattered if } \sigma(t) &> t, \\ t \textit{ is left dense if } \rho(t) &= t, \\ t \textit{ is left scattered if } \rho(t) &< t. \end{aligned}$$

$t \in \mathbb{T}$ is said to be a dense point if it is both right and left dense and $t \in \mathbb{T}$ is isolated point if it is both right and left scattered. The graininess functions $\mu : \mathbb{T} \to [0, \infty)$ and $\nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$ respectively.

The trivial continuous and discrete time scales $\mathbb{R}$ and $\mathbb{Z}$, have the following classifications:

If $\mathbb{T} = \mathbb{R}$, then for each $t \in \mathbb{R}$,

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t,$$

and similarly

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t.$$

Therefore each point of $\mathbb{R}$ is a dense point. And $\mu(t) \equiv \nu(t) \equiv 0$ for all $t \in \mathbb{R}$.
If $\mathbb{T} = \mathbb{Z}$, for each $t \in \mathbb{Z}$,

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, \cdots\} = t+1$$

and similarly

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = \sup\{\cdots, t-2, t-1\} = t-1.$$

Therefore each point of $\mathbb{Z}$ is an isolated point. And $\mu(t) \equiv \nu(t) \equiv 1$ for all $t \in \mathbb{Z}$.

2.1.2 Derivative on Time Scale

The region of Δ differentiability $\mathbb{T}^\kappa$, which is derived from $\mathbb{T}$ is required in order to define "*well-defined*" Δ-derivative. In literature, $\mathbb{T}^\kappa$ is called *Hilger's above truncated set* and is defined as follows [18]:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{\max \mathbb{T}\}, & \text{if } \max(\mathbb{T}) < \infty \text{ and } \max \mathbb{T} \text{ is left scattered,} \\ \mathbb{T} \quad , & \text{otherwise.} \end{cases}$$

Definition 2.1.2 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided that it exists) with the property that given $\varepsilon > 0$, there exists a neighborhood N_t of t (i.e., $N_t = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \forall s \in N_t. \tag{2.1}$$

We call $f^\Delta(t)$ the Δ- (or Hilger) derivative of f at t. Moreover, we say that f is Δ- (or Hilger) differentiable on $\mathbb{T}^\kappa$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Then the function $f^\Delta : \mathbb{T} \to \mathbb{R}$ is called the Δ-derivative of f on $\mathbb{T}^\kappa$.

Now we state the primary theorems on Δ- derivative on time scales.

Theorem 2.1.3 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) *If f is Δ-differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right scattered, then f is Δ-differentiable at t and*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right-dense, then f is Δ-differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(vi) *If f is Δ-differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Proof. See Theorem 1.16, (pp. 6-7) of [18]. □

Theorem 2.1.4 *Assume that the functions $f,\ g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^{\kappa}$. Then for all $\alpha,\ \beta \in \mathbb{R}$, we have the followings:*

(i) *The linear sum of f and g is Δ-differentiable at t and*

$$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t).$$

(ii) *The product $(fg) : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t and*

$$\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) \\ &= f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t)). \end{aligned}$$

(iii) *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ Δ-differentiable at t and*

$$(\frac{f}{g})^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))}.$$

Proof. See Theorem 1.20, (pp. 8-9) of [18]. $\square$

Next we state the ∇-analogues of the above definitions and theorems. The region of ∇-differentiability $\mathbb{T}_\kappa$, which is derived from $\mathbb{T}$ is required in order to define "*well-defined*" ∇-derivative [18].

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} - \{\min \mathbb{T}\}, & \text{if } \min(\mathbb{T}) > -\infty \text{ and } \min \mathbb{T} \text{ is right scattered,} \\ \mathbb{T} \qquad\qquad\quad, & \text{otherwise.} \end{cases}$$

Definition 2.1.5 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}_\kappa$. Then we define $f^\nabla(t)$ to be the number (provided that it exists) with the property that given $\varepsilon > 0$, there exists a neighborhood N_t of t such that*

$$|f(\rho(t)) - f(s) - a[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|, \quad \forall s \in N_t. \tag{2.2}$$

We call $f^\nabla(t)$ the ∇-derivative of f at t. Moreover, we say that f is ∇-differentiable on $\mathbb{T}_\kappa$ provided that $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$. Then the function $f^\nabla : \mathbb{T} \to \mathbb{R}$ is called the ∇-derivative of f on $\mathbb{T}_\kappa$.

Theorem 2.1.6 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}_\kappa$. Then we have the followings:*

(i) *If f is ∇-differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is left scattered, then f is ∇-differentiable at t and*

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

(iii) *If t is left dense, then f is ∇-differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists. In this case

$$f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(vi) *If f is ∇-differentiable at t, then*

$$f(\rho(t)) = f(t) - \nu(t) f^\nabla(t).$$

Proof. See Theorem 3.2, (pp. 47-48) of [20]. $\square$

Theorem 2.1.7 *Assume that the functions* $f,\ g : \mathbb{T} \to \mathbb{R}$ *are* ∇*-differentiable at* $t \in \mathbb{T}_\kappa$*. Then for all* $\alpha,\ \beta \in \mathbb{R}$*, we have the followings:*

(i) *The linear sum of* f *and* g *is* ∇*-differentiable at* t *and*

$$(\alpha f + \beta g)^\nabla(t) = \alpha f^\nabla(t) + \beta g^\nabla(t).$$

(ii) *The product* $(fg) : \mathbb{T} \to \mathbb{R}$ *is* ∇*-differentiable at* t *and*

$$\begin{aligned} (fg)^\nabla(t) &= f^\nabla(t)g(t) + f(\rho(t))g^\nabla(t) \\ &= f(t)g^\nabla(t) + f^\nabla(t)g(\rho(t)). \end{aligned}$$

(iii) *If* $g(t)g(\rho(t)) \neq 0$*, then* $\dfrac{f}{g}$ ∇*-differentiable at* t *and*

$$(\frac{f}{g})^\nabla(t) = \frac{f^\nabla(t)g(t) - f(t)g^\nabla(t)}{g(t)g(\rho(t))}.$$

Proof. See Theorem 3.3, (pp. 48) of [20]. $\square$

2.1.3 Integration on Time Scale

After Hilger introduced the basics of integration theory, the Riemann Δ- and ∇-integrability are constructed by Guseinov and Kaymakcalan [41]. Then Aulbach and Neidhard [15] improved the theory. The improper integral was initiated by Bohner and Guseinov [19]. The Lebesgue integral on time scales was introduced by Guseinov [42]. Cabada and Vivero [22] studied the relationship between Riemann and Lebesgue integrals on time scales. The fundamental integration theory can be found in almost each article related to time scales. These information are collected in the books [18, 20] by Bohner and Peterson.

In this subsection, we express some basic definitions and theorems which we need in the rest of the thesis.

Definition 2.1.8 [18] *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called rd-continuous provided it is continuous at all right dense points in* $\mathbb{T}$ *and its left-sided limits exist (finite)*

at all left dense points in $\mathbb{T}$. In this thesis, the set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by C_{rd}.
Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at all left dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all left dense points in $\mathbb{T}$. In this thesis, the set of ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by C_{ld}.

Theorem 2.1.9

(i) *Every rd-continuous function has a Δ-antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(s)\Delta s \text{ for } t \in \mathbb{T}$$

is a Δ-antiderivative of f.

(ii) *Every ld-continuous function has a ∇-antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(s)\nabla s \text{ for } t \in \mathbb{T}$$

is a ∇-antiderivative of f.

Proof. See Theorem 1.74, (pp. 27) and Theorem 8.45 (pp. 332) of [18]. □
Next we present two examples, Δ and ∇-integral on $\mathbb{Z}$, $\mathbb{R}$, and $h\mathbb{Z}$, to emphasize the difference between the ordinary derivative and integral and time scale integrals. The following theorem is very useful to evaluate the definite integral on discrete sets.

Theorem 2.1.10 *If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_{t}^{\sigma(t)} f(s)\Delta s = \mu(t)f(t),$$

and similarly if $f \in C_{ld}$ and $t \in \mathbb{T}_{\kappa}$, then

$$\int_{\rho(t)}^{t} f(s)\nabla s = \nu(t)f(t).$$

Proof. See Theorem 1.75, (pp. 28) and Theorem 8.46 (pp. 332) of [18]. □

Theorem 2.1.11 *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$. If $[a, b]$ consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & , \quad a < b; \\ 0 & , \quad a = b; \\ -\sum_{t\in[b,a)} \mu(t)f(t) & , \quad a > b. \end{cases}$$

Proof. See Theorem 5.37, (pp. 139-140) of [20]. □

We briefly summarized the Δ- and the ∇- integrals on time scales. The reader can find the detailed theory in the books [18, 20].

Example 2.1.12 *We consider the function $f(t) = t^2$on an arbitrary time scale. Let $\mathbb{T}$ be an arbitrary time scale and $t \in \mathbb{T}^k$.*

1. *If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$. Therefore $f^\Delta(t) = f'(t) = 2t$ and*
$$\int_a^b t^2\Delta t = \int_a^b t^2 dt.$$
2. *If $\mathbb{T} = \mathbb{Z}$ then $\sigma(t) = t + 1$. Therefore $f^\Delta(t) = \Delta f(t) = 2t + 1$ and*
$$\int_a^b t^2\Delta t = \sum_{n=a}^{b-1} n^2.$$
3. *If $\mathbb{T} = 2\mathbb{Z}$ then $\sigma(t) = t + 2$. Therefore $f^\Delta(t) = 2t + 2$ and*
$$\int_a^b t^2\Delta t = 2\sum_{n=a}^{b-2} n^2.$$

Similarly, to find ∇-derivative and integral, let us take $t \in \mathbb{T}^k$. The ∇-derivative and integral of f.

1. *If $\mathbb{T} = \mathbb{R}$ then $\rho(t) = t$. Therefore $f^\nabla(t) = f'(t) = 2t$ and*
$$\int_a^b t^2\nabla t = \int_a^b t^2 dt.$$

2. *If* $\mathbb{T} = \mathbb{Z}$ *then* $\rho(t) = t - 1$. *Therefore* $f^{\nabla}(t) = \nabla f(t) = 2t - 1$ *and*

$$\int_a^b t^2 \nabla t = \sum_{n=a}^{b+1} f(n).$$

3. *If* $\mathbb{T} = 2\mathbb{Z}$ *then* $\rho(t) = t - 2$. *Therefore* $f^{\nabla}(t) = 2t - 2$ *and*

$$\int_{2a}^{2b} t^2 \nabla t = 2 \sum_{n=2a}^{2b+2} f(n).$$

2.1.4 Measure Theory on Time Scale

The measure theory on time scales' content has two parts, Δ-measure and ∇-measure. In this subsection, we mention the basic definitions and theorems of Lebesgue Δ- and Lebesgue ∇-measure.
The measure theory on time scales is introduced by Guseinov [42]. The reader can find this article in the book [20] which is the collection of selected works. After that the theory is improved by Cabada and Vivero [22].

Let $\mathfrak{F}_1$ be the family of all left closed, right open intervals of a time scale $\mathbb{T}$. The set function $m_1 : \mathfrak{F}_1 \to [0, \infty]$ defined by

$$m_1([a, b)) = b - a$$

is a *countable additive measure* on $\mathfrak{F}_1$.

Let K be an arbitrary subset of a time scale $\mathbb{T}$ and the family $\{V_j \in \mathfrak{F}_1 : j \in \mathbb{N}\}$ be a finite or countable cover for K. We define the *outer measure* on $\mathbb{T}$ by

$$m_1^*(K) = \inf \sum_j m_1(V_j).$$

If such a finite or countable cover can not be found for K, then we set $m_1^*(K) = \infty$.

Definition 2.1.13 *A subset A of $\mathbb{T}$ is said to be m_1^* measurable if for all $K \subset \mathbb{T}$ the property*

$$m_1^*(K) = m_1^*(K \cap A) + m_1^*(K \cap A^C)$$

holds. Here A^C is the complement of A in $\mathbb{T}$ and is defined by $A^C = \mathbb{T} \setminus A$. The set of all m_1^* measurable sets is denoted by $\mathfrak{M}(m_1^*)$.

Definition 2.1.14 *The restriction of the outer measure m_1^* on $\mathfrak{M}(m_1^*)$ is called Lebesgue Δ-measure and denoted by μ_Δ.*

Lebesgue Δ-measure μ_Δ is endowed with the following properties:

- All the intervals belonging to $\mathfrak{F}_1$ are Δ-measurable.
- Since $\emptyset = [a, a)$, the empty set is Δ-measurable.
- The time scale $\mathbb{T}$ is Δ-measurable.

The single point sets has a Lebesgue measure zero on $\mathbb{R}$. Although this kind of sets may have nonzero Lebesgue Δ-measure on a time scale. The main difference Lebesgue Δ-measure and Lebesgue measure is this situation. The following theorem states this result.

Theorem 2.1.15 *If $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, then the single point set $\{t_0\}$ is Δ-measurable and*

$$\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0.$$

Proof. See Theorem 5.76, (pp. 158) of [20]. □

By making use of the previous theorem the Lebesgue Δ-measure of all kinds of intervals can be defined as follows:

Theorem 2.1.16 *If $a, b \in \mathbb{T}$ and $a \leq b$, then*

$$\mu_\Delta([a, b)) = b - a \quad \text{and} \quad \mu_\Delta((a, b)) = b - \sigma(a).$$

If $a, b \in \mathbb{T} \setminus \max \mathbb{T}$ and $a \leq b$, then

$$\mu_\Delta((a, b]) = \sigma(b) - \sigma(a) \quad \text{and} \quad \mu_\Delta([a, b]) = \sigma(b) - a.$$

Proof. See Theorem 5.77, (pp. 158-159) of [20]. □

By denoting the family of all left open, right closed intervals of a time scale by $\mathfrak{F}_2$, the set function $m_2 : \mathfrak{F}_2 \to [0, \infty]$,

$$m_2((a, b]) = b - a$$

can be defined on this family. Thus the outer measure m_2^*, and the Lebesgue ∇-measure μ_∇ can be defined in a similar manner. For more detailed information see [20, 42].

The ∇ analogues of the Theorems 2.1.15 and 2.1.16 are as follows:

Theorem 2.1.17 *If $t_0 \in \mathbb{T} \setminus \{\min \mathbb{T}\}$, then the single point set $\{t_0\}$ is ∇-measurable and*

$$\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0).$$

Proof. See Theorem 5.78, (pp. 159) of [20]. □

Theorem 2.1.18 *If $a, b \in \mathbb{T}$ and $a \leq b$, then*

$$\mu_\nabla((a, b)) = b - a \quad \text{and} \quad \mu_\nabla((a, b)) = \rho(b) - a.$$

If $a, b \in \mathbb{T} \setminus \min \mathbb{T}$ and $a \leq b$, then

$$\mu_\nabla([a, b)) = \rho(b) - \rho(a) \quad \text{and} \quad \mu_\nabla([a, b]) = b - \rho(a).$$

Proof. See Theorem 5.79, (pp. 159) of [20]. □

The reader can find the more details about the the Lebesgue Δ- and ∇-measures and the relationships between these measures and the Lebesgue Δ- and ∇- integrals in Guseinov [42] and Cabada and Vivero [22].

2.1.5 Time Scale Calculus on Banach Spaces

In this subsection, we deal with the generalizations of the definitions of Δ- and ∇-derivatives and Δ- and ∇-integrals for Banach valued functions. Also we mention and prove the mean value theorems for Δ- and ∇-integrals. For more details, for the calculus for Banach space valued function in discrete and continuous case see [21, 16, 4, 26], for time scale case see [29, 30, 31, 59].

Definition 2.1.19 *Let E be a Banach space, $u : \mathbb{T} \to E$ be a function and $t \in \mathbb{T}^\kappa$. Then we define $u^\Delta(t)$ to be the number (provided that it exists) with the property* *that given $\varepsilon > 0$, there exists a neighborhood N_t of t (i.e., $N_t = (t-\delta, t+\delta) \cup \mathbb{T}$ for some $\delta > 0$) such that*

$$|u(\sigma(t)) - u(s) - u^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \forall s \in N_t. \tag{2.3}$$

Definition 2.1.20 *Let E be a Banach space, $u : \mathbb{T} \to E$ be a function and $t \in \mathbb{T}_\kappa$. Then we define $u^{\nabla}(t)$ to be the number (provided that it exists) with the property that given $\varepsilon > 0$, there exists a neighborhood N_t of t (i.e., $N_t = (t-\delta, t+\delta) \cup \mathbb{T}$ for some $\delta > 0$) such that*

$$|u(\rho(t)) - u(s) - u^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|, \quad \forall s \in N_t. \tag{2.4}$$

Definition 2.1.21 *Assume that $U^{\Delta}(t) = u(t)$ for $t \in \mathbb{T}^\kappa$. The Δ-integral on a Banach space is defined by*

$$\int_a^t u(\tau)\Delta\tau = U(t) - U(a),$$

while under the assumption $W^{\nabla}(t) = w(t)$ for $t \in \mathbb{T}_\kappa$, the ∇-integral on a Banach space is

$$\int_a^t w(\tau)\nabla\tau = W(t) - W(a).$$

2.1.6 Partial Derivatives on Time Scales

In this section, we present a brief introduction to multivariable time scale calculus. The multivariable calculus of time scales is created by Ahlbrandt and Morian, [23] and Jackson [17] in order to study the partial dynamic equations. We improve these definitions and theorems for Banach valued functions.

Consider the product $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$ where $\mathbb{T}_i$ is a time scale for all $1 \leq i \leq n$. Then for any $t \in \mathbb{T}$, with $t = (t_1, t_2, \cdots, t_n)$ for $t_i \in \mathbb{T}_i$ for all $1 \leq i \leq n$ define the following :

- the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

  $\sigma(t) = (\sigma(t_1), \sigma(t_2), \cdots, \sigma(t_n))$, where $\sigma(t_i)$ represents the forward jump operator of $t_i \in \mathbb{T}_i$ on the time scale $\mathbb{T}_i$ for all $1 \leq i \leq n$. Hereafter, the forward jump operator of the time scale $\mathbb{T}_i$ for $t_i \in \mathbb{T}_i$ will be denoted by $\sigma(t_i) = \sigma_i(t)$.

- the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

  $\rho(t) = (\rho(t_1), \rho(t_2), \cdots, \rho(t_n))$, where $\rho(t_i)$ represents the backward jump operator of $t_i \in \mathbb{T}_i$ on the time scale $\mathbb{T}_i$ for all $1 \le i \le n$. Hereafter, the forward jump operator of the time scale $\mathbb{T}_i$ for $t_i \in \mathbb{T}_i$ will be denoted by $\rho(t_i) = \rho_i(t)$.

- the graininess function $\mu : \mathbb{T} \to \mathbb{R}^n$ by

  $\mu(t) = (\mu(t_1), \mu(t_2), \cdots, \mu(t_n))$, where $\mu(t_i)$ represents the backward jump operator of $t_i \in \mathbb{T}_i$ on the time scale $\mathbb{T}_i$ for all $1 \le i \le n$. Again, from this point on the graininess function of the time scale $\mathbb{T}_i$ for $t_i \in \mathbb{T}_i$ will be denoted by $\mu(t_i) = \mu_i(t)$.

- $\mathbb{T}^k = \mathbb{T}_1^k \times \mathbb{T}_2^k \times \cdots \times \mathbb{T}_n^k$.

Until here, we have defined the multivariate time scale forward jump operator, now we will define the partial Δ derivative of a function $f(t)$. Before doing this, we must give some other notations. From here on, set

$f^{\sigma_i}(t) = f(t_1, t_2, \cdots, t_{i-1}, \sigma_i(t), t_{i+1}, \cdots, t_n)$ and the set

$f_i^s(t) = f(t_1, t_2, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n)$ (i.e. to evaluate $f_i^s(t)$ replace t_i in $f(t)$ by s.)

Definition 2.1.22 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t = (t_1, t_2, \cdots, t_n) \in \mathbb{T}^k$. Then define $f^{\Delta_i}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t_i, with $U = (t_i - \delta, t_i + \delta) \cap \mathbb{T}_i$ for $\delta > 0$ such that*

$$\left|[f^{\sigma_i}(t) - f_i^s(t)] - f^{\Delta_i}(t)[\sigma_i(t) - s]\right| \le \varepsilon \left|\sigma_i(t) - s\right| \text{ for all } s \in U.$$

f^{Δ_i} is called the partial delta derivative of f at t with respect to the variable t_i.

Among to these definitions, it can be understood that to find the partial derivative with respect to t_i, the other variables must be seen as constants with respect to

t_i and taking the usual derivative of $f(t)$ in the t_i variable on the time scale $\mathbb{T}_i$. Thus, the definition is same as the generalization of its continuous analog, which follows from the fact that if $\mathbb{T}_i = \mathbb{R}$ for all i, then the partial delta derivative is the usual continuous partial derivative. In the same way, if $\mathbb{T}_i = h\mathbb{Z}$ for all i, then the partial delta derivative is the known partial difference operator as given in [62]. With these investigations, it can be seen that $f^{\Delta_{ij}}(t)$ (if this value exists) is found by first taking the partial derivative with respect to t_i to get $f^{\Delta_i}(t)$, and then taking the partial derivative of this derivative function with respect to t_j to obtain $f^{\Delta_{ij}}(t)$, so that $f^{\Delta_{ij}} = (f^{\Delta_i})^{\Delta_j}$. Higher order mixed partials are defined and evaluated by the same way. To evaluate $f^{\Delta_{iii\cdots i}}(t)$ where i occurs n times, the other notion that will be used is to take the partial derivative of $f(t)$ with respect to t_i n times. From the disputation about mixed partials that we have mentioned above, it can be easily understood that evaluating this derivative is equivalent to evaluating $f^{\Delta_i^n}(t)$, where Δ_i^n denotes taking the delta derivative with respect to t_i on the time scale $\mathbb{T}_i$ n times. In order to simplfy the expressions, we use $f^{\Gamma\Delta}(x, y)$ for the second order partial derivative of $f : \mathbb{T} \to \mathbb{R}$ with respect to x and y respectively.

2.2 Weak Time Scale Calculus

The weak solutions of Cauchy differential equation, was studied by Szep [61] for the first time. Later on the theorems on the existence of weak solutions of Cauchy problem were proved by Cramer, Lakshmikantham, and Mitchell [33], Kubiaczyk [48], Mitchell and Smith [55], Szufla [47] Cichon [27], Cichon and Kubiaczyk [26]. In Banach spaces for solving existence problems for difference equations with similar methods equipped with its weak topology were studied, for instance in [1, 34, 51]. Weak time scale calculus is created by [29] and improved by [63]. We improve these theory for multivariable calculus in order to construct and obtain the existence of weak solutions of the hyperbolic Cauchy problem in Banach spaces.

Definition 2.2.1 *A function $f : \mathbb{T} \to E$ is said to be weakly continuous if it is continuous from $\mathbb{T}$ to E, endowed with its weak topology. A function $g : E \to$*

E_1, where E and E_1 are Banach spaces, is said to be weakly-weakly sequentially continuous if, for each weakly convergent sequence (x_n) in E, the sequence $(g(x_n))$ is weakly convergent in E_1. When the sequence x_n tends weakly to x_0 in E, we write $x_n \overset{w}{\to} x_0$.

Definition 2.2.2 [29] *We say that $u : \mathbb{T} \to E$ is weakly right dense continuous (weakly rd-continuous) if u is weakly continuous at every right dense point $t \in \mathbb{T}$ and exists and $\lim_{s \to t^-} u(s)$ is finite at every left dense point $t \in \mathbb{T}$.*

The so-called Δ and ∇-weak derivative and Δ, ∇-weak integral for Banach valued functions are defined by generalizing the notions Δ-derivative and Δ- integral on time scales [18, 20].

Definition 2.2.3 [29] *Let $u : \mathbb{T} \to E$. Then we say that u is Δ-weak differentiable at $t \in \mathbb{T}$ if there exists an element $U(t) \in E$ such that for each $x^* \in E^*$ the real valued function x^*u is Δ-differentiable at t and $(x^*u)^{\Delta}(t) = (x^*U)(t)$. Such a function U is called Δ-weak derivative of u and denoted by $u^{\Delta w}$.*

For the ∇ analogue we can give similar definition.

Definition 2.2.4 *Let $u : \mathbb{T} \to E$. Then we say that u is ∇-weak differentiable at $t \in \mathbb{T}$ if there exists an element $U(t) \in E$ such that for each $x^* \in E^*$ the real valued function x^*u is ∇-differentiable at t and $(x^*u)^{\nabla}(t) = (x^*U)(t)$. Such a function U is called ∇-weak derivative of u and denoted by $u^{\nabla w}$.*

Definition 2.2.5 [29] *If $U^{\Delta w} = u(t)$ for all t, then we define Δ-weak Cauchy integral by*

$$C\!\!\!\!\!\!\!\!\!\quad w\!\!\!\int_a^t u(\tau)\Delta\tau = U(t) - U(a).$$

The ∇-weak Cauchy integral can be defined in a similar way:

Definition 2.2.6 *If $U^{\nabla w} = u(t)$ for all t, then we define ∇-weak Cauchy integral by*

$$C\!\!\!\!\!\!\!\!\!\quad w\!\!\!\int_a^t u(\tau)\nabla\tau = U(t) - U(a).$$

By generalizing the Theorem 1.74 of [18], the existence of weak antiderivatives can be obtained:

Definition 2.2.7 [29]*(Existence of Δ-weak antiderivatives). Every weakly rd-continuous function has a weak antiderivative. In particular if $t_0 \in \mathbb{T}$ then U defined by*

$$U(t) := C\!\!\!\!\int_{t_0}^{t} u(\tau)\Delta\tau,\ t \in \mathbb{T}$$

is a weak Δ antiderivative of u.

Definition 2.2.8 *(Existence of ∇-weak antiderivatives). Every weakly ld-continuous function has a weak antiderivative. In particular if $t_0 \in \mathbb{T}$ then U defined by*

$$U(t) := C\!\!\!\!\int_{t_0}^{t} u(\tau)\nabla\tau,\ t \in \mathbb{T}$$

is a weak ∇ antiderivative of u.

Since the Δ-weak (∇-weak) Cauchy integral is defined by means of weak antiderivatives, the space of Δ-weak, (∇-weak) Cauchy integrable functions (i.e. the space of weakly rd-continuous functions) is too narrow. Therefore, we need to define the Δ-weak and ∇-weak Riemann integral for Banach space-valued function.

Definition 2.2.9 [29] *Let $P = \{a_0, a_1, \cdots, a_n\}$ be a partition of the interval $[a, b]$. P is called $finer$ than $\delta > 0$ either*

- *$\mu_\Delta([a_{i-1}, a_i)) \le \delta$ or*
- *$\mu_\Delta([a_{i-1}, a_i)) > \delta$ if only $a_i = \sigma(a_{i-1})$.*

Definition 2.2.10 [29] *A function $u : [a, b] \to E$ is called Δ-weak Riemann integrable if there exists $U \in E$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: For any partition $P = \{a_0, a_1, \cdots, a_n\}$ which is finer than δ and any set of points $t_1, t_2, \cdots, t_n$ with $t_j \in [a_{j-1}, a_j)$ for $j = 1, 2, \cdots, n$ one has*

$$\left| x^*(U) - \sum_{j=1}^{n} x^*(u(t_j))\mu_\Delta([a_{j-1}, a_j)) \right| \le \varepsilon, \quad \forall x^* \in E^*.$$

According to Definition 2.1.19, U is uniquely determined and it is called the Δ-weak Riemann integral of u and denoted by

$$U = Rw\int_a^b u(t)\Delta t.$$

Definition 2.2.11 *A function $u : [a, b] \to E$ is called weak ∇-Riemann integrable if there exists $U \in E$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: For any partition $P = \{a_0, a_1, \cdots, a_n\}$ which is finer than δ and any set of points $t_1, t_2, \cdots, t_n$ with $t_j \in [a_{j-1}, a_j)$ for $j = 1, 2, \cdots, n$ one has*

$$\left| x^*(U) - \sum_{j=1}^{n} x^*(u(t_j))\mu_{\nabla}([a_{j-1}, a_j)) \right| \le \varepsilon, \quad \forall x^* \in E^*.$$

According to Definition 2.1.20, U is uniquely determined and it is called the ∇-weak Riemann integral of u and denoted by

$$U = Rw\int_a^b u(t)\nabla t.$$

By regarding the definitions of weak integrals and by using Theorem 4.3 of Guseinov [42], we are able to state that every Riemann Δ and ∇-weak integrable function is a Cauchy Δ and ∇-weak integrable and in this case, these two integrals coincide.

Theorem 2.2.12 [29]*(Mean value theorem for Δ-weak integrals) If the function $f : \mathbb{T} \to E$ is Δ- weak integrable then*

$$Rw\int_{I_b} f(t)\Delta t \in \mu_{\Delta}(I_b) \cdot \overline{conv} f(I_b),$$

where I_b is an arbitrary subinterval of the time scale $\mathbb{T}$ and $\mu_{\Delta}(I_b)$ is the Lebesgue Δ- measure of I_b.

Proof. See Theorem 2.11, (pp.4-5) of [29]. □

Similar to the preceeding theorem we construct the mean value theorem for ∇-weak integrals.

Theorem 2.2.13 *(Mean Value Theorem for ∇-weak integrals) If the function $f : \mathbb{T} \to E$ is ∇- weak integrable then*

$$Rw\int_{I_b} f(t)\nabla t \in \mu_\nabla(I_b) \cdot \overline{conv} f(I_b),$$

where I_b is an arbitrary subinterval of the time scale $\mathbb{T}$ and $\mu_\nabla(I_b)$ is the Lebesgue ∇- measure of I_b.

Proof. Put $v = Rw\int_A y(s)\nabla s$ and $W = \mu_\nabla(A) \cdot \overline{conv}\ y(A)$. Suppose to the contrary that $v \notin W$. By the separation theorem for convex sets, there exists $z^* \in E^*$ such that $\sup_{x\in W} z^*(x) = \alpha < z^*(v)$. But

$$z^*(v) = z^*\left(Rw\int_A y(s)\nabla s\right) = \int_A z^*(y(s))\nabla s.$$

Moreover since $y(s) \in y(A)$, for all $s \in A$, we have

$$\mu_\nabla(A) \cdot y(s) \in \mu_\nabla(A) \cdot \overline{conv}\ y(A) = W, \text{ i.e., } \quad y(s) \in \frac{1}{\mu_\nabla(A)} \cdot W.$$

Thus $z^*(y(s)) \le \frac{1}{\mu_\nabla(A)} \cdot \alpha$. Finally we obtain

$$z^*(v) = \int_A z^*(y(s))\nabla s \le \int_A \frac{\alpha}{\mu_\nabla(A)}\nabla s = \frac{\alpha}{\mu_\nabla(A)}.\mu_\nabla(A) = \alpha$$

which is a contradiction. □

See [18, 20, 22, 42, 15] and for the definition and basic properties of the Lebesgue Δ-measure and the Lebesgue Δ-integral.

In order to study hyperbolic partial dynamic problem stated in Chapter 4, we need to define the Riemann double integrability and the mean value theorem for double integrals.

Definition 2.2.14 *(Riemann Double Integrability) A Banach space valued-function $f : [a, b] \times [c, d] \to E$ is called weak Riemann double integrable if there exists $U \in E$ such that for any $\varepsilon > 0$ there exists a δ with the following property: For any partition $\mathcal{P}_1 = \{a_0, a_1, \cdots, a_n\}$ of $[a, b]$ and $\mathcal{P}_2 = \{c_0, c_1, \cdots, c_n\}$ of $[c, d]$ which are finer than δ and the set of points $x_j \in [a_{j-1}, a_j)$ and $y_j \in [c_{j-1}, c_j)$ for $j = 1, 2, \cdots, n$ one has*

$$\left| x^*(U) - \sum_{j=1}^{n} x^*(f(x_j, y_j))\mu_\Delta([a_{j-1}, a_j) \times [c_{j-1}, c_j)) \right| \le \ \varepsilon, \forall x^* \in E^*.$$

The function U is called weak Riemann double integral f and denoted by

$$U = Rw\int_a^b Rw\int_c^d f(x,y)\Delta y\Gamma x.$$

Theorem 2.2.15 *(Mean Value Theorem for Double Integrals) If the function $h : \mathbb{T}_1 \times \mathbb{T}_2 \to E$ is Δ and Γ-weak integrable,then*

$$\iint_{\mathcal{R}} h(x,y)\Delta y\Gamma x \in \mu_\Delta(\mathcal{R}) \cdot \overline{conv}\ h(\mathcal{R})$$

where $\mathcal{R}$ is an arbitrary subset of $\mathbb{T}_1 \times \mathbb{T}_2$.

Proof. Let $\iint_{\mathcal{R}} h(x,y)\Delta y\Gamma x = v$ and $\mu_\Delta(\mathcal{R}) \cdot \overline{conv}\ h(\mathcal{R}) = W$. Suppose to the contrary, that $v \notin W$. By separation theorem for the convex sets there exists $z^* \in E^*$ such that

$$\sup_{g\in W} z^*(g) = \alpha < z^*(v),$$

then

$$z^*(v) = z^*\left(Cw\int Cw\int_{\mathcal{R}} h(x,y)\Delta y\Gamma x\right) = \iint z^*(h(x,y))\Delta y\Gamma x.$$

And let $(s,t) \in \mathcal{R}$, we have $h(s,t) \in h(\mathcal{R})$. We get

$$\mu_\Delta(\mathcal{R}).h(s,t) \subseteq \mu_\Delta(\mathcal{R}).\overline{conv}\ h(\mathcal{R}) = W$$

$$h(s,t) \subseteq \frac{W}{\mu_\Delta(\mathcal{R})}$$

implies

$$z^*(h(s,t)) \le z^*\left(\frac{W}{\mu_\Delta(\mathcal{R})}\right) < \frac{\alpha}{\mu_\Delta(\mathcal{R})}.$$

Finally we obtain,

$$z^*(v) = \iint z^*(h(s,t))\Delta y\Gamma x \le \iint_{\mathcal{R}} \frac{\alpha}{\mu_\Delta(\mathcal{R})}\Delta y\Gamma x = \frac{\alpha}{\mu_\Delta(\mathcal{R})} \cdot \mu_\Delta(\mathcal{R}) = \alpha$$

which is a contradiction. □

2.3 Basic Definitions and Theorems

In this section, we present the main theorems and the definitions which will be applied throughout the dissertation.

Theorem 2.3.1 [7] *Let E be a Banach space and $M \subset E$. M is relatively compact if the following conditions hold:*

1. *M is uniformly bounded in E,*
2. *The functions taken from M are equicontinuous on any compact interval of $[0, \infty)$,*
3. *The functions taken from M are equiconvergent, i.e., for any given ε there exists a real number $T = T(\varepsilon) > 0$ such that $|f(t) - f(\infty)| < \varepsilon$, for any $t > T,\ f \in M$.*

Definition 2.3.2 [32] *An operator is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Definition 2.3.3 [32] *Let $\mathcal{F}$ be the family of functions from the metric space (X, d) to the metric space (X', d'). The family $\mathfrak{F}$ is uniformly equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F},\ x, y \in X$*

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon.$$

More generally, when X is a topological space, a set $\mathfrak{F}$ of functions from X to X' is said to be equicontinuous at x if for every $\varepsilon > 0$, x has a neighborhood N_x such that $d'(f(y), f(x)) < \varepsilon$ for all $y \in N_x$ and $f \in \mathfrak{F}$. This definition usually appears in the context of of topological vector spaces.

Theorem 2.3.4 (Arzelà-Ascoli Theorem) [57] *Let Ω be a bounded subset of R^n and (f_k) be the sequence of the functions from Ω to R^m. If (f_k) is equicontinuous and uniformly equibounded then there exists a uniformly convergent subsequence (f_{k_j}) of (f_k).*

Definition 2.3.5 [32] *Let X be a metric space and $A \subseteq X$. A is said to be totally bounded if there exist a a finite subset $\{s_1, s_2, \ldots, s_n\}$ of A such that $A \subseteq \bigcup_{k=1}^{n} B(s_k, \varepsilon)$ for $\varepsilon > 0$ where $B(s_k, \varepsilon)$ denotes the open ball with center s_k and radius ε.*

The more abstract (more general) version of Arzelà-Ascoli Theorem is as folows:

Theorem 2.3.6 (Arzelà-Ascoli Theorem)[32] *Let X and Y be totally bounded metric spaces and $F \subset C(X, Y)$ be the family of uniformly equicontinuous functions. Then F is totally bounded with respect to uniformly convergence metric generated by $C(X, Y)$.*

We note that the first version of Arzelà-Ascoli Theorem is a result of the second one. Really, a complete metric space is totally bounded if and only if its closure is compact. Thus Ω is totally compact and the images of each f_k are in a totally bounded set. The totally compactness of $F = \{f_k\}$ implies the compactness of $\overline{F}$. Hence (f_k) has a convergent subsequence.

Definition 2.3.7 [32] *Let $\mathcal{F}$ be the family of functions from the metric space X and the metric space Y. If there exists a bounded subset B of Y such that $f(x) \in B$ for all $f \in \mathcal{F}$ and $x \in X$, then the family $\mathcal{F}$ is said to be equibounded.*

Note that, if $\mathcal{F} \subset C_b(X, Y)$ (the set of continuous and bounded functions), then F is equibounded if and only if $\mathcal{F}$ is bounded with respect to uniformly boundedness metric.

Definition 2.3.8 [57] *Let Ω be a convex set on $\mathbb{R}$ and $f : \Omega \to \mathbb{R}$. If*

$$f\Big(\lambda x + (1-\lambda)y\Big) \geq \lambda f(x) + (1-\lambda)f(y)$$

hold for all $x, y \in \Omega$ and $\lambda \in (0,1)$, then the function f is said to be a concave function.

Theorem 2.3.9 [58]*(Sadovskii Fixed Point Theorem) If $F : B \to B$ is contiuous mapping satisfying $\phi(F(V)) < \phi(V)$ for arbitrary nonempty subset V of B with $\phi(V) > 0$, then F has a fixed point in B.*

Theorem 2.3.10 [50]*(Kubiaczyk Fixed Point Theorem) Let X be a metrizable locally convex topological vector space. Let D be a closed convex subset of X, and let F be a sequentially continuous map from D into itself. If for some $x \in D$ the implication*

$$\overline{V} = \overline{conx}(\{x\} \cup F(V)) \Rightarrow V \quad \textit{relatively weakly compact.}$$

holds for every subset V of D, then F has a fixed point.

2.4 Measure of Weak Noncompactness

The measure of weak noncompactness was developed by De Blasi [36].Then it is used in numerous branches of functional analysis and the theory of differential and integral equations.
Let A be a bounded nonempty subset of Banach space E. The measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{ there exists } C \in K^w \text{ such that } A \subset C + tB_1\}$$

where K^w is the set of weakly compact subsets of E and B_1 is the unit ball in E. We will utilize the below properties of the measure of weak noncompactness β (for bounded nonempty subsets A and B of E):

1. If $A \subset B$ then $\beta(A) \leq \beta(B)$,
2. $\beta(A) = \beta(\overline{A}^w)$, where $\overline{A}^w$ denotes the weak closure of A,
3. $\beta(A) = 0$ if and only if A is relatively compact,
4. $\beta(A \cup B) = max\{\beta(A), \beta(B)\}$,
5. $\beta(\lambda A) = |\lambda|\, \beta(A)$ where $\lambda \in R$,
6. $\beta(A + B) \leq \beta(A) + \beta(B)$,
7. $\beta(\overline{conx}(A)) = \beta(conx(A)) = \beta(A)$, where $conv(A)$ denotes the convex hull of A.

The following lemma which is an adaptation of the corresponding result of Banaś and Goebel [16] is true, when β is an arbitrary set function that satisfies the above properties.

Lemma 2.4.1 [29] *If* $\|E_1\| = \sup\{\|x\| : x \in E_1\} < 1$ *then*

$$\beta(E_1 + E_2) \leq \beta(E_2) + \|E_1\|\, \beta(K(E_2, 1)),$$

where $K(E_2, 1) = \{x : d(E_2, 1) \leq 1\}$.

Lemma 2.4.2 [63] *Let X be an equicontinuous bounded set in $C(\mathbb{T}, E)$, where $C(\mathbb{T}, E)$ denotes the space of all continuous functions from the time scale $\mathbb{T}$ to the Banach space E.*
Denote

$$\int_0^a X(s)\Delta s = \left\{ \int_0^a x(s)\Delta s : x \in X \right\}.$$

$$\beta\left(\int_0^a X(s)\Delta s \right) \leq \int_0^a \beta(X(s))\Delta s.$$

Proof. For $\delta > 0$ we choose points in $\mathbb{T}$ in the following way:

$$\begin{aligned}
t_0 &= 0, \\
t_1 &= \sup_{s \in I_a} \{s : s \geq t_0, s - t_0 < \delta\}, \\
t_2 &= \sup_{s \in I_a} \{s : s > t_1, s - t_1 < \delta\}, \\
t_3 &= \sup_{s \in I_a} \{s : s > t_2, s - t_2 < \delta\}, \\
&\dots \\
t_{n-1} &= \sup_{s \in I_a} \{s : s > t_{n-2}, s - t_{n-2} < \delta\}, \\
t_n &= a.
\end{aligned}$$

If some $t_i = t_{i-1}$ then $t_{i+1} = \inf\limits_{s \in I_a} \{s : s > t_i\}$. By the equicontinuity of X there exists $\delta > 0$ and $\xi_i \in [t_{i-1}, t_i]$ such that

$$\left\| \int_0^a x(s)\Delta s - \sum_{i=1}^n x(\xi_i)\mu_\Delta(t_{i-1}, t_i) \right\| \leq \varepsilon.$$

Thus we have

$$\begin{aligned}
\int_0^a X(s)\Delta s \quad &\subset \quad \left\{ \int_0^a x(s)\Delta s - \sum_{i=1}^n x(\xi_i)\mu_\Delta(t_{i-1}, t_i) : \; x \in X \right\} \\
&\qquad + \left\{ \sum_{i=1}^n x(\xi_i)\mu_\Delta(t_{i-1}, t_i) : \; x \in X \right\} \\
&= \quad A + B.
\end{aligned}$$

Now

$$\beta(A) \le \beta(K(0,\varepsilon)) = \varepsilon\beta(K(0,1))$$

and

$$\beta(B) \le \sum_{i=1}^{n} \mu_\Delta(t_{i-1},t_i)\beta(X(\xi_i)).$$

Therefore

$$\beta\left(\int_0^a X(s)\Delta s\right) \le \beta(A+B) \le \varepsilon\beta(K(0,1)) + \sum_{i=1}^{n} \mu_\Delta(t_{i-1},t_i)\beta(X(\xi_i)).$$

If $\varepsilon \to 0$ and $n \to \infty$ we obtain

$$\beta\left(\int_0^a X(s)\Delta s\right) \le \int_0^a \beta(X(s))\Delta s.$$

□

The lemma below is an adaptation of the corresponding result of Ambrosetti [12] and it is proved in [29].

Lemma 2.4.3 *Let $H \subset C(\mathbb{T},E)$ be a family of strongly equicontinuous functions. Let $H(t) = \{h(t) \in E, h \in H\}$, for $t \in \mathbb{T}$. Then*

$$\beta(H(\mathbb{T})) = \sup_{t\in\mathbb{T}} \beta(H(t)),$$

and the function $t \mapsto \beta(H(t))$ is continuous on $\mathbb{T}$.

Proof. See Lemma 2.9, (pp.4) of [29]. □

The generalization of Ambrosetti Lemma for $C(\mathbb{T}_1 \times \mathbb{T}_2, E)$ is as follows:

Lemma 2.4.4 *Let $H \subset C(\mathbb{T}_1 \times \mathbb{T}_2, E)$ be a family of strongly equicontinuous functions. Let $H(x,y) = \{h(x,y) \in E, h \in H\}$, for $(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2$. Then*

$$\beta(H(\mathbb{T}_1 \times \mathbb{T}_2)) = \sup_{(x,y)\in\mathbb{T}_1\times\mathbb{T}_2} \beta(H(x,y)),$$

and the function $(x,y) \mapsto \beta(H(x,y))$ is continuous on $\mathbb{T}_1 \times \mathbb{T}_2$.

Let us denote by S_∞ the set of all nonnegative real sequences. For $\xi = \xi_n \in S_\infty$, $\eta = \eta_n \in S_\infty$, we write $\xi < \eta$ if $\xi_n \le \eta_n$ (i.e. $\xi_n \le \eta_n$, for $n = 1, 2, \cdots$) and $\xi \ne \eta$.

Let B be a closed convex subset of $(C(\mathbb{T}, E), W)$ and ϕ be a function which assigns to each nonempty subset V of B, a sequence $\phi(V) \in S_\infty$, such that

$$\phi(\{x\} \cup V) = \phi(V), \quad \text{for} \quad x \in B, \tag{2.5}$$

$$\phi(\overline{conv}\ V) = \phi(V), \tag{2.6}$$

$$\text{if } \phi(V) = \emptyset \text{ (the zero sequence) then } \overline{V} \text{ is compact.} \tag{2.7}$$

Chapter 3

Cauchy Dynamic Equation

A dynamic equation is a differential equation, difference equation, quantum equation or more interestingly a combination of all these. The study of dynamic equations on time scales is an area of mathematics research that has recently received a lot of attention. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics.

In this chapter, we prove the existence of weak solutions of second order Cauchy dynamic problem

$$\begin{array}{l} x^{\Delta\nabla}(t) = f(t, x(t)), \\ x(0) = 0, \quad x^{\Delta}(0) = \eta_1 \end{array}, \quad t \in \mathbb{T}, \quad \eta_1 \in E \tag{3.1}$$

in Banach spaces.

The first order Cauchy difference equation $\Delta x(t) = f(t, x(t))$ and the first order Cauchy differential equation $x'(t) = f(t, x(t))$ in Banach spaces have been studied may many authors in the literature. Authors obtained the conditions expressed by measure of (weak) noncompactess for the existence of classical solutions, Carathéodory solutions, weak solutions and pseudo solutions. In the articles [55, 27, 29, 33, 61, 63] the properties of solution sets are presented.

The study on the dynamic equations in Banach spaces is started with the landmark article by Cichon *et.al.* [27]. The existence of weak solutions for the first order dynamic Cauchy problem over an unbounded time scale is presented. Authors remark that the condition on the nonlinear term given in terms of measure of noncompactness can be generalized to Szufla condition or Sadowskii condition, also measure of weak noncomapctness can be replaced by any axiomatic measure of noncompactness. The study dynamic equation in Banach spaces is continued by [29] and [63].

3.1 Equivalent Integral Operator

In this section, first we express the problem (3.1) as an integral equation. Then we write the equivalent integral operator corresponding to the problem (3.1). For constructing the equivalent integral operator we set the followings.
Let $L^1(\mathbb{T})$ denote the space of real valued Δ- Lebesgue integrable functions on a time scale $\mathbb{T}$. Assume that there exists a function $M \in L^1(\mathbb{T}), M(t) \geq 0, t \in \mathbb{T}$, such that $\|f(t,x)\| \leq M(t)\ \mu_\Delta$ a.e. on $\mathbb{T}$, for some $x \in E$.
Let

$$b_t = \|\eta_1\|\, t + Cw\!\!\int_0^t Cw\!\!\int_0^{t_1} M(t_2)\nabla t_2 \Delta t_1, \tag{3.2}$$

$$K(\tau, s) = Cw\!\!\int_\tau^s Cw\!\!\int_0^{t_1} M(t_2)\nabla t_2 \Delta t_1, \tag{3.3}$$

$$p(t) = \eta_1 . t, \tag{3.4}$$

$$\begin{aligned}\widetilde{B}_t = \{x \in C(I_t, E) : \|x(t)\| \leq b_t, &\|x(\tau) - x(s)\| \leq \|p(\tau) - p(s)\| + K(\tau, s),\\ &t, \tau, s \in \mathbb{T},\quad 0 \leq s < \tau < t\}\end{aligned} \tag{3.5}$$

where $\mathbb{T}$ denotes an unbounded time scale and $I_t = \{s \in T : 0 \leq s \leq t\}$.

We recall that a function $g : E \to E$ is a weakly-weakly sequentially continuous if $x_n \xrightarrow{w} x_0$ in E then $g(x_n) \xrightarrow{w} g(x_0)$ in E.

Definition 3.1.1 *A function $x : \mathbb{T} \to E$ is said to be a weak solution of the problem* (3.1) *if x has Δ-weak derivative, x^{Δ} has ∇-weak derivative and satisfies* (3.1) *for all $t \in \mathbb{T}$.*

We consider an appropriate integral equation

$$x(t) = \eta_1 \cdot t + Cw\int_0^t Cw\int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1 \tag{3.6}$$

We note that each solution of the problem (3.6) is the solution of (3.1) and converse. Now we verify the equivalence of (3.6) and (3.1). For this purpose assume that a weakly continuous function $x : \mathbb{T} \to E$ is a weak solution of (3.1). By using the definition of ∇-weak integrals (Definition 2.2.8), we have

$$\begin{aligned} Cw\int_0^{t_1} f(t_2, x(t_2))\nabla t_2 &= Cw\int_0^{t_1} x^{\Delta\nabla}(t_2)\nabla t_2 \\ &= x^{\Delta}(t_1) - x^{\Delta}(0), \end{aligned}$$

that is,

$$x^{\Delta}(t_1) = \eta_1 + Cw\int_0^{t_1} f(t_2, x(t_2))\nabla t_2. \tag{3.7}$$

Hence

$$Cw\int_0^t x^{\Delta}(t_1)\Delta t_1 = Cw\int_0^t \eta_1 \Delta t_1 + Cw\int_0^t Cw\int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1$$

Applying the Δ-weak integral (Definition 2.2.7), we obtain

$$x(t) - x(0) = \eta_1 . t + Cw\int_0^t Cw\int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1$$

which means that the function x satisfies the integral equation (3.6). Next, let the function x be the solution of the integral equation (3.6). For any $x^* \in E^*$, we have

$$(x^* x)(t) = x^*(\eta_1 . t + \int_0^t \int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1)$$

Using the definition of Δ-weak derivative (Definition 2.2.3), we get

$$\begin{aligned} (x^* x)^{\Delta}(t) &= x^* \left(\eta_1 . t + \int_0^t \int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1 \right)^{\Delta} \\ &= [x^*(\eta_1 . t)]^{\Delta} + [\int_0^t \int_0^{t_1} x^*(f(t_2, x(t_2))\nabla t_2 \Delta t_1)]^{\Delta} \\ &= x^*(\eta_1) + \int_0^t x^*(f(t_2, x(t_2)))\nabla t_2 \end{aligned}$$

Next we apply ∇-weak derivative (Definition 2.2.4):

$$\begin{aligned}(x^*x)^{\Delta\nabla}(t) &= [x^*(\eta_1) + \int_0^t x^*(f(t_2, x(t_2)))\nabla t_2]^{\nabla} \\ &= (x^*(\eta_1))^{\nabla} + \left(\int_0^t x^*(f(t_2, x(t_2)))\nabla t_2\right)^{\nabla} \\ &= x^*(f(t_2, x(t_2))),\ \forall x^* \in E^*.\end{aligned}$$

Therefore $x^{\Delta\nabla}(t) = f(t, x(t))$ in weak sense. Now, we show that the initial conditions of (3.1) hold.

$$x(0) = \eta_1.0 + C\!\!\!\!\int_0^0 C\!\!\!\!\int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1 = 0,$$

i.e. the first initial condition holds. Using equation (3.7), we show that x also satisfies second initial condition.

$$x^{\Delta}(0) = \eta_1 + C\!\!\!\!\int_0^0 f(t_2, x(t_2))\nabla t_2 = \eta_1$$

Therefore $x(t)$ is the solution of (3.1), *i.e.* the Cauchy problem (3.1) and the integral equation (3.6) are equivalent.

Let the operator $F : (C(\mathbb{T}, E); w) \to (C(\mathbb{T}, E); w)$ be defined by

$$F(x)(t) = p(t) + C\!\!\!\!\int_0^t C\!\!\!\!\int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1 \qquad (3.8)$$

where $p(t)$ was defined in (3.4). By the considerations above, finding a solution for the problem (3.1) is equivalent to finding a fixed point of the integral operator (3.8).

3.2 Existence Result

In this section, we prove the existence of weak solution for the Cauchy problem (3.1) by proving the existence of fixed points of the corresponding integral operator (3.8) using Theorem 2.3.9. We state the sufficient conditions-as possible as close to necessary conditions- by means of DeBlasi measure of weak noncompactness β. The condition given in terms of measure of weak noncompactness can be generalized to Sadowskii of Szufla conditions. Also measure of weak noncompactness can be replaced by any axiomatic measure of noncompactness.

Theorem 3.2.1 *Let $f : \mathbb{T} \times E \to E$ be a function and suppose that a and b bounded and integrable functions from $C_{rd}(\mathbb{T}, \mathbb{R})$ such that*

$$\|f(t,x)\| \leq a(t) + b(t)\,\|x(t)\| \tag{3.9}$$

for each $(t,x) \in (\mathbb{T}, E)$. Moreover, let the following conditions hold:

(C1) $f(t,\cdot)$ is weakly-weakly sequentially continuous, for each $t \in \mathbb{T}$;

(C2) for each strongly absolutely continuous function $x : \mathbb{T} \to E$; $f(\cdot, x(\cdot))$ is weakly continuous,

(C3) there exists a function $L : \mathbb{T} \times [0,\infty) \to [0,\infty)$ such that for each continuous function $u : [0,\infty) \to [0,\infty)$ the mapping $t \mapsto L(t,u)$ is continuous and $L(t,0) \equiv 0$ on $\mathbb{T}$,

(C4) for all $r > 0$

$$\int_0^\infty \int_0^{t_1} L(t_2, r)\nabla t_2 \Delta t_1 < r,$$

(C5) for any compact subinterval I of $\mathbb{T}$ and each nonempty bounded subset A of E

$$\beta(f(I_b \times A)) \leq \sup_{t \in I_b} L(t, \beta(A)).$$

Then there exists at least one Δ-weak solution of the problem (3.1) *on some subinterval $I_b \subset \mathbb{T}$.*

Proof. The condition $C2$ implies that the operator $F : \widetilde{B}_t \to (C(\mathbb{T}, E), w)$ is well-defined. Now we show that the operator F maps $\widetilde{B}_t \to \widetilde{B}_t$.

- First we verify $\|F(x)(t)\| \leq b_t$.

$$\begin{aligned}
\|F(x)(t)\| &= \left\| p(t) + \int_0^t \int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1 \right\| \\
&\leq \|p(t)\| + \left\| \int_0^t \int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1 \right\| \\
&\leq \|\eta_1\|\, t + \left\| \int_0^t \int_0^{t_1} f(t_2, x(t_2))\nabla t_2 \Delta t_1 \right\| \\
&\leq \|\eta_1\|\, t + \int_0^t \int_0^{t_1} \|f(t_2, x(t_2))\|\, \nabla t_2 \Delta t_1 \\
&\leq \|\eta_1\|\, t + \int_0^t \int_0^{t_1} \|M(t_2)\|\, \nabla t_2 \Delta t_1 = b_t
\end{aligned}$$

- Consequently we show, the set $F(\widetilde{B}_t)$ is almost equicontinuous.

$$\begin{aligned}
\|F(x)(\tau)-F(x)(s)\| = \|p(\tau) &+ C\!\!\!\!\!\!\!\int_0^{\tau} C\!\!\!\!\!\!\!\int_0^{t_1} f(t_2,x(t_2))\nabla t_2\Delta t_1 - p(s) \\
&- C\!\!\!\!\!\!\!\int_0^{s} C\!\!\!\!\!\!\!\int_0^{t_1} f(t_2,x(t_2))\nabla t_2\Delta t_1\| \\
&\le \|p(\tau)-p(s)\| + \left\|\int_s^{\tau}\int_0^{t_1} f(t_2,x(t_2))\nabla t_2\Delta t_1\right\| \\
&\le \|p(\tau)-p(s)\| + \int_s^{\tau}\int_0^{t_1} \|f(t_2,x(t_2))\|\nabla t_2\Delta t_1 \\
&\le \|p(\tau)-p(s)\| + \int_s^{\tau}\int_0^{t_1} M(t_2)\nabla t_2\Delta t_1 \\
&\le \|p(\tau)-p(s)\| + K(\tau,s)
\end{aligned}$$

where $\tau, s \in \mathbb{T}$ and $x \in \widetilde{B}_t$. Hence $F(\widetilde{B}_t)$ is strongly almost equicontinuous.

- Now we show weakly sequentially continuity of the integral operator F.

Let $x_n \xrightarrow{w} x$ in $\widetilde{B}_t$. Fix an arbitrary $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for $n \ge N$ and each $t \in I_\alpha$, we have $|x^*x_n(t) - x^*x(t)| < \varepsilon$. From condition (C1), we get

$$|x^*f(t_2,x_n(t_2)) - x^*f(t_2,x(t_2))| < \frac{\varepsilon}{\alpha^2}.$$

Therefore

$$\begin{aligned}
|x^*(F(x_n)(t)-F(x)(t))| &= \left|x^*\left(\int_0^t\int_0^{t_1} f(t_2,x_n(t_2))\nabla t_2\Delta t_1 - \int_0^t\int_0^{t_1} f(t_2,x(t_2))\nabla t_2\Delta t_1\right)\right| \\
&\le \int_0^t\int_0^{t_1} |x^*f(t_2,x_n(t_2)) - x^*f(t_2,x(t_2))|\nabla t_2\Delta t_1 \\
&\le \int_0^t\int_0^{\alpha} |x^*f(t_2,x_n(t_2)) - x^*f(t_2,x(t_2))|\nabla t_2\Delta t_1 \\
&< \int_0^t \alpha\frac{\varepsilon}{\alpha^2}\Delta t_1 = \int_0^t \frac{\varepsilon}{\alpha}\Delta t_1 \le \int_0^{\alpha}\frac{\varepsilon}{\alpha}\Delta t_1 = \varepsilon.
\end{aligned}$$

From items it was shown that F is well-defined, weakly sequentially continuous and maps $\widetilde{B}_t$ into $\widetilde{B}_t$.
It can be observed that the weak solution of the problem (3.1), is the fixed point of operator F. Now we'll prove that a fixed point of the operator F can be

obtained by using the fixed point theorem (Theorem 2.3.9). Let V be a subset of $\widetilde{B}_t$ satisfying the condition

$$\overline{V} = \overline{conv}(\{x\} \cup F(V)) \ for\ some\ x \in \widetilde{B}_t.$$

We prove that V is relatively weakly compact.
The functions $a(t)$, $b(t)$ satisfying the sublinearity condition (3.9) can be chosen in a way that

$$\int_{\xi}^{\infty} |a(s)| \nabla s + \int_{\xi}^{\infty} |b(s)| \, \|x(s)\| \nabla s < \varepsilon.$$

We divide the interval $[0, \xi]$ into m parts.

$$0 = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = \xi$$

such that the partition is finer than δ (Definition 2.2.9). We define the subinterval

$$T_i = [t_i, t_{i+1}] \cap \mathbb{T}$$

and

$$V(T_i) = \{x(s) \in E : x \in V \text{ and } s \in T_i\}.$$

By the definition of $F(x)(t)$, T_i, and the mean value theorem (Theorem 2.2.12) we have

$$\begin{aligned}
F(x)(t) &= p(t) + \int_0^t \int_0^{t_1} f(t_2, x(t_2)) \nabla t_2 \Delta t_1 \\
&= p(t) + \int_0^t \left(\int_0^{\xi} f(t_2, x(t_2)) \nabla t_2 + \int_{\xi}^{t_1} f(t_2, x(t_2)) \nabla t_2 \right) \Delta t_1 \\
&= p(t) + \int_0^t \left(\sum_{i=0}^{m-1} \int_{T_i} f(t_2, x(t_2)) \nabla t_2 + \int_{\xi}^{t_1} f(t_2, x(t_2)) \nabla t_2 \right) \Delta t_1 \\
&\in p(t) + \int_0^t \left(\sum_{i=0}^{m-1} \mu_{\nabla}(T_i) \cdot f(T_i, V(T_i)) + \int_{\xi}^{t_1} f(t_2, x(t_2)) \nabla t_2 \right) \Delta t_1
\end{aligned}$$

for each $x \in V$. Hence

$$F(V)(t) \subset p(t) + \int_0^t \left(\sum_{i=0}^{m-1} \mu_{\nabla}(T_i) \cdot f(T_i \times V(T_i)) + \int_{\xi}^{t_1} f(t_2, x(t_2)) \nabla t_2 \right) \Delta t_1.$$

Using (C5),(C4), the properties of measure of weak noncompactness β, Lemma 2.4.1 and Lemma 2.4.2, we obtain

$$\begin{aligned}
\beta(F(V)(t)) &\leq \beta\left(p(t) + \int_0^t \left(\sum_{i=0}^{m-1} \mu_\nabla(T_i)\cdot f(T_i \times V(T_i)) + \int_\xi^{t_1} f(t_2, V(t_2))\nabla t_2\right)\Delta t_1\right) \\
&\leq \int_0^t \beta\left(\sum_{i=0}^{m-1} \mu_\nabla(T_i)\cdot f(T_i \times V(T_i))\right) + \left(\left\|\int_\xi^{t_1} f(t_2, V(t_2))\nabla t_2\right\|\right)\Delta t_1 \\
&\leq \int_0^t \left(\sum_{i=0}^{m-1} \mu_\nabla(T_i)\cdot \beta\left(f(T_i \times V(T_i))\right) + \sup_{x\in V}\int_\xi^{t_1} f(t_2, x(t_2))\nabla t_2\right)\Delta t_1 \\
&\leq \int_0^t \left(\sum_{i=0}^{m-1} \mu_\nabla(T_i)\cdot \sup_{\tau_i\in T_i} L(\tau_i, \beta(V(T_i))) + \int_\xi^{\infty} a(t_2) + b(t_2)\left\|x(t_2)\right\|\nabla t_2\right)\Delta t_1 \\
&\leq \int_0^t \left(\sum_{i=0}^{m-1} \mu_\nabla(T_i)\cdot L(\tau, \beta(V(T_i))) + \varepsilon\right)\Delta t_1
\end{aligned}$$

where τ is defined in a way that $L(\tau,\beta(V(K))) = \sup_{\tau_i\in K}\{L(\tau_i,(V(K)))\}$. Since ε is arbitrary we get

$$\begin{aligned}
\beta(F(V)(t)) &\leq \int_0^t \int_0^{t_1} \left(L(s, \beta(V[0,\xi]))\nabla s\right)\Delta t_1 \\
&\leq \int_0^\infty \int_0^{t_1} \left(L(s, \beta(V[0,\xi]))\nabla s\right)\Delta t_1 < \beta(V(0,\xi)) < \beta(V(0,t)),
\end{aligned}$$

for $\beta(V(t)) \geq 0$.

If $\beta(V(t)) = 0$ then by the properties of measure of weak noncompactness, V is relatively weakly compact. Thus the assumptions of fixed point theorem of Kubiaczyk (Theorem 2.3.10) are fulfilled. Hence F has a fixed point which is the solution of the problem (3.1).

We define $\phi(V(t)) = \beta(V(t))$, it is evident that $\phi(F(V(t))) < \phi(V(t))$ whenever $\phi(V) > 0$. It can be seen that all assumptions of Sadovskii's fixed point theorem (Theorem (2.3.9)) have been satisfied, F has a fixed point in $\widetilde{B}_t$ *i.e.* the Cauchy problem (3.1) has a solution and the proof is complete. $\square$

Chapter 4

Cauchy Partial Dynamic Equation

In this chapter we stated an existence result for the weak solutions of the hyperbolic Cauchy problem (HCP)

$$\begin{array}{l} z^{\Gamma\Delta}(x,y) = f(x,y,z(x,y)), \\ z(x,0) = 0, \quad z(0,y) = 0 \end{array}, \quad x \in \mathbb{T}_1, \quad y \in \mathbb{T}_2 \tag{4.1}$$

where the nonlinear term $f : \mathbb{T}_1 \times \mathbb{T}_2 \times E \to E$ and $z : \mathbb{T}_1 \times \mathbb{T}_2 \to E$. The differential operators Γ and Δ are taken in weak sense.

The study on the classical solutions of the hyperbolic Cauchy differential equation $z_{xy}(x,y) = f(x,y,z(x,y))$ is initiated by Davidowski, Kubiaczyk and Rzepecki [35]. Authors expressed the sufficient conditions in terms of Kuratowski measure of noncompactness to guarantee the existence of solutions. In this chapter we improve the results of this article in two different aspects: We construct the problem for the general case, *i.e.* time scale case and optimize the conditions. Also we consider the weak solutions.

4.1 Equivalent Integral Operator

In this section, first we obtain an integral equation which is equivalent to (4.1). Then we assign the integral equation to an integral operator satisfying the conditions of Sadovskii and Kubiaczyk fixed point theorems.

We remark that a weakly continuous function $z : \mathbb{T}_1 \times \mathbb{T}_2 \to E$ is said to be a *weak solution* of HPC (4.1) if z has Γ-weak partial derivative, z^Γ has Δ-weak partial derivative and satisfies (4.1) for all $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$.

We claim that in the case of weakly-weakly continuous f, finding a weak solution of HPC (4.1) is equivalent to solving the integral equation

$$z(x,y) = \int_0^x \int_0^y f(u,v,z(u,v))\Delta v \Gamma u, \quad (x,y) \in \mathbb{T}_1 \times \mathbb{T}_2. \tag{4.2}$$

Here integrals are considered in weak sense.

Assume that a weakly continuous function $z : \mathbb{T}_1 \times \mathbb{T}_2 \to E$ is a weak solution of the HCP (4.1). We show that z solves the integral equation (4.2). By the definition of weak Cauchy integral (Definition 2.2.5), we have

$$\begin{aligned} \int_0^y f(x,v,z(x,v))\Delta v &= \int_0^y z^{\Gamma\Delta}(x,v)\Delta v \\ &= z^\Gamma(x,y) - z^\Gamma(x,0) = z^\Gamma(x,y) \end{aligned}$$

Note that $z^\Gamma(x,0) = 0$ since $z(x,0) = 0$. If we integrate the resulting equation on $[0,x]_{\mathbb{T}_1}$, we obtain

$$\begin{aligned} \int_0^x \int_0^y f(u,v,z(u,v))\Delta v \Gamma u &= \int_0^x z^\Gamma(u,y)\Gamma u \\ &= z(x,y) - z(0,y) = z(x,y) \end{aligned}$$

which implies that z solves the integral equation (4.2).

Conversely, we assume that $z(x, y)$ is a solution of the integral equation (4.2). For any $z^* \in E^*$, we have

$$(z^* z)(x,y) = z^* \left(\int_0^x \int_0^y f(u,v,z(u,v))\Delta v \Gamma u \right)$$

and therefore

$$\begin{aligned}(z^*z)^\Gamma &= \left(\int_0^x \int_0^y z^*(f(u,v,z(u,v)))\Delta v \Gamma u\right)^\Gamma \\ &= \int_0^y z^*(f(x,v,z(x,v)))\Delta v.\end{aligned}$$

Differentiating the last expression we get

$$\begin{aligned}(z^*z)^{\Gamma\Delta} &= \left(\int_0^y z^*(f(x,v,z(x,v)))\Delta v\right)^\Delta \\ &= z^*(f(x,y,z(x,y))).\end{aligned}$$

By the definition of weak partial derivatives (Definition 2.1.22), we obtain

$$z^{\Gamma\Delta}(x,y) = f(x,y,z(x,y)).$$

Clearly the boundary conditions of (4.1) hold. Hence $z(x,y)$ is a solution of (4.1).

We consider the space of continuous functions with its weak topology, *i.e.*,

$$(C(\mathbb{T}_1 \times \mathbb{T}_2, E), w) = (C(\mathbb{T}_1 \times \mathbb{T}_2, E), \tau(C(\mathbb{T}_1 \times \mathbb{T}_2, E), C^*(\mathbb{T}_1 \times \mathbb{T}_2, E)))\,.$$

By the equivalence of (4.1) and (4.2), the fixed points of the integral operator $F : (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w) \to (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w)$

$$F(z)(x,y) = \int_0^x \int_0^y f(u,v,z(u,v))\Delta v \Gamma u, \quad (x,y) \in \mathbb{T}_1 \times \mathbb{T}_2 \tag{4.3}$$

are the weak solutions of the HCP (4.1)

4.2 Existence of Weak Solutions

In this section the existence of a weak solution of HCP (4.1) is obtained by applying Sadovskii and Kubiaczyk fixed point theorems to the corresponding integral operator (4.3). The conditions on the nonlinear term f in the main result is stated in terms of measure of weak noncompactness. The mean value theorem for double integrals (Theorem 2.2.15) developed by generalizing the result of [29] and the Ambosetti's Lemma is used to prove the main result.

Let $G : \mathbb{T}_1 \times \mathbb{T}_2 \times [0, \infty) \to [0, \infty)$ be continuous function and nondecreasing in the last variable. Assume that the scalar integral inequality

$$g(x, y) \geq \int_0^x \int_0^y G(u, v, ||z(u, v)||)\Delta v \Gamma u \tag{4.4}$$

has locally bounded solution $g_0(x, y)$ existing on $\mathbb{T}_1 \times \mathbb{T}_2$.

We define the nonempty, closed, bounded, convex and equicontinuous function set $X \subset (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w)$ as follows:

$$\begin{aligned} X = \Big\{ & z \in (C(\mathbb{T}_1 \times \mathbb{T}_2, E), w) : ||z(x, y)|| \leq g_0(x, y), \text{ on } \mathbb{T}_1 \times \mathbb{T}_2 \\ & ||z(x_1, y_1) - z(x_2, y_2)|| \leq \left| \int_0^{x_2} \int_{y_1}^{y_2} G(u, v, g_0(u, v))\Delta v \Gamma u \right| \\ & + \left| \int_{x_1}^{x_2} \int_0^{y_1} G(u, v, g_0(u, v))\Delta v \Gamma u \right| \text{ for } x_1, x_2 \in \mathbb{T}_1 \text{ and } y_1, y_2 \in \mathbb{T}_2 \Big\} \end{aligned} \tag{4.5}$$

Theorem 4.2.1 *Let $L : \mathbb{T}_1 \times \mathbb{T}_2 \times [0, \infty) \to [0, \infty)$ be a function such that for each $u \in [0, \infty)$ the mapping $(x, y) \mapsto L(x, y, u)$ is continuous and $L(x, y, 0) \equiv 0$ on $\mathbb{T}_1 \times \mathbb{T}_2$. Moreover, let the following condition hold:*

(D1) f is weakly weakly sequentially continuous for each $x \in \mathbb{T}_1$ and $y \in \mathbb{T}_2$,

(D2) $||f(x, y, u)|| \leq G(x, y, ||u||)$ for $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $u \in E$,

(D3) $\beta(f(P \times W)) \leq \sup\{L(x, y, \beta(W)) : (x, y) \in P\}$ for any compact subset P of $\mathbb{T}_1 \times \mathbb{T}_2$ and each nonempty bounded subset W of E,

(D4) $\int_0^\infty \int_0^\infty L(u, v, r)\Delta v \Gamma u \leq r$ for all $r > 0$.

Then there exists a solution of (4.1) satisfying

$$||z(x, y)|| \leq g_0(x, y) \quad \textit{for } (x, y) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Proof. In order to use fixed point theorems, we first show that F maps X to X. Using the conditions (D2), (D4) and the inequality (4.4) respectively, we get

$$\begin{aligned} ||F(z)(x, y)|| &= \left|\left| \int_0^x \int_0^y f(u, v, z(u, v))\Delta v \Gamma u \right|\right| \\ &\leq \int_0^x \int_0^y ||f(u, v, z(u, v))||\Delta v \Gamma u \\ &\leq \int_0^x \int_0^y G(u, v, ||z(u, v)||)\Delta v \Gamma u \leq g_0(x, y). \end{aligned} \tag{4.6}$$

Consequently we show that

$$\begin{aligned} ||F(z)(x_1,y_1)-F(z)(x_2,y_2)|| &\leq \left|\int_0^{x_2}\int_{y_1}^{y_2} G(u,v,g_0(u,v))\Delta v\Gamma u\right| \\ &\quad + \left|\int_{x_1}^{x_2}\int_0^{y_1} G(u,v,g_0(u,v))\Delta v\Gamma u\right| \end{aligned}$$

For simplicity in the following calculations, let f stand for $f(u,v,z(u,v))$.

$$\begin{aligned} \|F(z)(x_1,y_1)-F(z)(x_2,y_2)\| &= \left\|\int_0^{x_1}\int_0^{y_1} f\Delta v\Gamma u - \int_0^{x_2}\int_0^{y_2} f\Delta v\Gamma u\right\| \\ &= \left\|\int_0^{x_2}\int_0^{y_1} f\Delta v\Gamma u + \int_{x_2}^{x_1}\int_0^{y_1} f\Delta v\Gamma u - \int_0^{x_2}\int_0^{y_2} f\Delta v\Gamma u\right\| \\ &= \left\|\int_0^{x_2}\left(\int_0^{y_1} f\Delta v - \int_0^{y_2} f\Delta v\right)\Gamma u + \int_{x_2}^{x_1}\int_0^{y_1} f\Delta v\Gamma u\right\| \\ &= \left\|\int_0^{x_2}\int_{y_2}^{y_1} f\Delta v\Gamma u + \int_{x_2}^{x_1}\int_0^{y_1} f\Delta v\Gamma u\right\| \\ &= \left\|\int_0^{x_2}\int_{y_1}^{y_2} f\Delta v\Gamma u + \int_{x_1}^{x_2}\int_0^{y_1} f\Delta v\Gamma u\right\| \\ &\leq \left\|\int_0^{x_2}\int_{y_1}^{y_2} f\Delta v\Gamma u\right\| + \left\|\int_{x_1}^{x_2}\int_0^{y_1} f\Delta v\Gamma u\right\| \\ &\leq \left|\int_0^{x_2}\int_{y_1}^{y_2} G(u,v,g_0(u,v))\Delta v\Gamma u\right| + \left|\int_{x_1}^{x_2}\int_0^{y_1} G(u,v,g_0(u,v))\Delta v\Gamma u\right| \end{aligned}$$

Therefore, F maps X to X.

Next we show weakly-sequentially continuity of the integral operator (4.3). Let $z_n \overset{w}{\to} z$ in X. Fix an arbitrary $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $n \geq N$ and each $t \in I_\alpha \times I_\alpha \subset \mathbb{T}_1 \times \mathbb{T}_2$, we have $|z^* z_n(x,y) - z^* x(x,y)| < \varepsilon$. From condition (D1), it follows that

$$|z^* f(x,y,z_n(x,y)) - z^* f(x,y,z(x,y))| < \frac{\varepsilon}{\alpha^2}.$$

Therefore we lead

$$\begin{aligned}
|z^*(F(z_n)(x,y) - F(z)(x,y))| &= \Big|z^*\Big(\int_0^x \int_0^y f(u,v,z_n(u,v))\Delta v\Gamma u \\
&\qquad - \int_0^x \int_0^y f(u,v,z_n(u,v))\Delta v\Gamma u\Big)\Big| \\
&\leq \int_0^x \int_0^y |z^* f(u,v,z_n(u,v)) - z^* f(u,v,z(u,v))| \,\Delta v\Gamma u \\
&\leq \int_0^\alpha \int_0^\alpha |z^* f(u,v,z_n(u,v)) - z^* f(u,v,z(u,v))| \,\Delta v\Gamma u \\
&\leq \int_0^\alpha \int_0^\alpha \frac{\varepsilon}{\alpha^2}\Delta v\Gamma u = \varepsilon.
\end{aligned}$$

Since $\mathbb{T}_1 \times \mathbb{T}_2$ is a closed subset of $\mathbb{R}^+ \times \mathbb{R}^+$, it is locally compact Hausdorf space. By the result of Dobrakov [38], F is weakly-sequentially continuous mapping. From the considerations above, F is well-defined, weakly-sequentially continuous and maps X to X.

Let $f(x,y,z(x,y))$ satisfy the sublinearity condition

$$\|f(x,y,z(x,y))\| \leq a(x,y) + b(x,y)\,\|z(x,y)\|$$

for each $(x,y,z) \in \mathbb{T}_1 \times \mathbb{T}_2 \times E$, where $a(x,y)$ and $b(x,y)$ are bounded and integrable functions taken from $C(\mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R})$. Since $a(x,y)$ and $b(x,y)$ are integrable functions, we have

$$\iint_R |a(u,v)|\,\Delta v\Gamma u + \iint_R |b(u,v)|\,\|z(u,v)\|\,\Delta v\Gamma u < \varepsilon \tag{4.7}$$

where $R = \mathbb{T}_1 \times \mathbb{T}_2 - ([0,\xi] \times [0,\eta])$. We divide $[0,\xi] \subset \mathbb{T}_1$ into m parts and $[0,\eta] \subset \mathbb{T}_2$ into m parts

$$0 < x_1 < x_2 < x_3 < \cdots < x_m = \xi$$

$$0 < y_1 < y_2 < y_3 < \cdots < y_m = \eta$$

such that each partition is finer than δ (Definition 2.2.9).
We define $T_1^i = [x_i, x_{i+1}] \cap \mathbb{T}_1$ and $T_2^j = [y_j, y_{j+1}] \cap \mathbb{T}_2$. Then there exists $(\sigma_i, \tau_j) \in T_1^i \times T_2^j = P_{ij}$ such that

$$\beta(W(\sigma_i,\tau_j)) = \beta(W(P_{ij})) = \sup\{\beta(W(x,y)) : (x,y) \in P_{ij}\}$$

Using mean value theorem for double integrals (Theorem 2.2.15) we lead

$$\begin{aligned} F(z)(x,y) &= \int_0^x \int_0^y f(u,v,z(u,v))\Delta v \Gamma u \\ &= \int_0^\xi \int_0^\eta f(u,v,z(u,v))\Delta v \Gamma u + \iint_R f(u,v,z(u,v))\Delta v \Gamma u \\ &\in \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_\Delta(P_{ij}) \cdot \overline{conv}(P_{ij} \times W(P_{ij})) + \iint_R f(u,v,z(u,v))\Delta v \Gamma u \end{aligned}$$

for all $z \in W$. Therefore

$$F(W)(x,y) \subset \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_\Delta(P_{ij}) \cdot \overline{conv}(P_{ij} \times W(P_{ij})) + \iint_R f(u,v,z(u,v))\Delta v \Gamma u. \tag{4.8}$$

Consequently we apply β to each sides of equation (4.8) and we get

$$\begin{aligned} \beta(F(W)(x,y)) &\leq \sup \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_\Delta(P_{ij}) L(x,y,\beta(W)) + \left\| \iint_R f(u,v,z(u,v))\Delta v \Gamma u \right\| \\ &\leq \sup \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mu_\Delta(P_{ij}) L(x,y,\beta(W)) + \varepsilon \\ &\leq \int_0^x \int_0^y L(u,v,\beta(W))\Delta v \Gamma u + \varepsilon \\ &\leq \beta(W)(x,y) \end{aligned}$$

for $\beta(W) \geq 0$.

If $\beta(W) = 0$ then using the properties of measure of weak of noncompactness, W is relatively weakly compact. Thus all the assumptions of fixed point theorem of Kubiaczyk are satisfied. Hence F has a fixed point which is the solution of our HPC (4.1).

We define $\phi(W) = \beta(W)$. If $\phi(W) > 0$ then $\phi(F(W)) < \phi(W)$. Hence all assumptions of Sadovski fixed point theorem are fulfilled, F has a fixed point *i.e.* the HPC (4.1) has a solution. The proof is complete. □

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