YASAR UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLED SCIENCE

MASTER THESIS

EXTREMUM PROBLEM WITH THE CONSTRAINTS

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I certifythat I havereadthisthesisandthat in myopinion it is fullyadequate, inscopeand in quality, as a dissertationforthedegree of master of science.

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ABSTRACT

EXTREMUM PROBLEM WITH THE CONSTRAINTS

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MSc in Mathematics
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In this thesis by using Dubovitskii Milyutin theorem we investigate necessary optimality condition for optimal control system. In this way we used general form of Euler equation and separate principle of conex cone.

The main idea of the method are generalizations ideas with which investigated problems on extremum with constraints in the case of functions of a finite number of variables.

Keywords: Necessary optimality conditions, The first variation, Supporting functionals, Linear convex functional, Convex set.

ÖZET

KISITLAMA ŞARTLI EKSTREMUM PROBLEMİ

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Bu tezde Dubovitskii Milyutin teoremini kullanarak optimal kontrol sistemi için gereklilik şarti araştırdık. Bunun için genel Euler denklemi ve koneks konların ayrılma prensibini kullandık.

Yöntemin temel amacı kısıtlama şartı olan optimal kontrol sestemleri için gerek koşulların bulunmasıdır.

Anahtar sözcükler: Gerekli optimalite koşulları, Birinci varyasyon, Destek fonksiyonları, Lineer konveks fonksiyonları, Konveks küme

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Finally, I extend my thanks and appreciations to my dear parents who never hesitated to supports me my life and especially during my studies. My dear wife for patience and continuous encouragement and my dear brother and sister to be always being there for me

for helping me always throughout my life and especially during my studies. I never forget their help. I also would like to thank my wife who supports and constant help and encourages me in life. Furthermore, many thanks for all my brothers and sisters, who always wishing.

TEXT OF OATH

I, Feridoon Saleh RASOOL do declare and honestly confirm that my study, titled ″ EXTREMUM PROBLEM WITH THE CONSTRAINTS″ and presented as a Master′s Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the bibliography, and that I have benefited from these sources by means of making references.

Student Name and Signature
Feridoon Saleh RASOOL

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INTRODUCTION

In this thesis by using Dubovitskii Milyutin theorem (1.1) we investigate necessary optimality condition for optimal control system. In this way we used general form of Euler equation and separate principle of conex cone. The discovery of Maximum Principle (MP) by L.S. Pontryagin and his students V.G. Boltyanskii and R.V. Gamkrelidze (1956-58), and especially the publication of the book by Pontryagin, Boltyanskii, Gamkrelidze and E.F. Mischchenko (1961), gave a powerful impetus to an explosive development of the theory both of the optimal control itself, and of extremum problems in general. The main idea of the method are generalizations ideas with which investigated problems on extremum with constraints in the case of functions of a finite number of variables. However, for the application of these ideas to variational problems is required a known amount of knowledge of the facts of functional analysis. Because the thesis is intended not only to mathematicians, then it is given a lot of space ”technique” with the required object function analysis. The main ”technical” question is how to investigate this or other specific restrictions and how to write a general form of the functional cone conjugate present. When ”solving” this problem, we went the following path. Since, according to the proposed method, the restriction or functionality of this type is sufficient to investigate only one once, and then simply transfer the results of the task in the task, we tried highlight the most typical and functional limitations and lead their detailed research. Here we omit the proofs theorems of functional analysis, which only serve to strictly justify the legality of certain actions.

The thesis consists of three chapters. In the first chapter we give some basic definitions, examples and theorems related to necessary optimality conditions contains a description of the method, shows the function theorem analysis that substantiates the method.

In the second chapter we give the definition of the general form of a linear functional from the cone conjugate the cone of a special kind. For convenience, the end of each of these sections Rules are formulated in the form of the main results.

In the study targets sufficiently mean only those rules. Relying on them, we solve a large number of examples in which collected the most common functional and cones. We believe that attentive acquaintance with all the examples is a necessary condition mastery of the proposed method .

In the third chapter, we present the solution a number of problems in the optimal regulation. Among these problems are the problem with constraints on the phase coordinates and minimal problem.

CHAPTER 1

NECESSARY OPTIMALITY CONDITIONS

In this chapter, we give some basic definitions, examples and theorems related to necessary optimality conditions, we will start a general approach to variational problem, with the help of that obtaining necessary optimality conditions. We always call the necessary conditions resulting from application of this method, the Euler equation. However, these conditions will not always be in the form of a differential equation, but connected only with the specific features of the problem, the method of obtaining the necessary conditions remains unchanged, and essentially summarizes a method for obtaining Euler equations in classical variational problems.

We will try to illustrate the main features of the method of the problem analysis, on those problems that to find extremum functionals of a finite number of variables under certain restrictions. The following information it can be find in the reference Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004). Let us, give a continuously differentiable functional of n variables $f(x_1, x_2, \ldots, x_n)$. It is required to find the extremum of this functional with restriction $\varphi(x_1, x_2, \ldots, x_n) = 0$, where some of φ is continuously differentiable functional and grad $\varphi \neq 0$, grad $f \neq 0$ in the area that we are interested in. Suppose that $(x_1{}^0, ..., x_n{}^0)$ is an extreme point. It is known that in this case according to the rules of Lagrange multipliers, necessary conditions as follows:

There is a number λ such that for the functional $H = f - \lambda\varphi$ at the point $(x_1{}^0, ..., x_n{}^0)$ all the partial derivatives vanish. This result can be obtained by the following geometric reviews. Let's say that the variation of $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ is prohibited variation, if $\varepsilon > 0$ and $\varepsilon \to 0$

$$\lim \frac{d}{d\varepsilon} f(x^0 + \varepsilon\bar{x}) < 0. \tag{1.1}$$

since

$$\frac{d}{d\varepsilon} f(x^0 + \varepsilon\bar{x}) = \frac{\partial f}{\partial x_1}\bar{x}_1 + \cdots + \frac{\partial f}{\partial x_n}\bar{x}_n \qquad if \qquad \varepsilon = 0,$$

where the partial derivatives are taken at the point $x = x^0$, then the set of all prohibited $\bar{x}$ is determined by a linear inequality with constant coefficients.

Next we call the variation $\bar{x}$ admissible to the restriction $\varphi = 0$ if $\varepsilon > 0$ and $\varepsilon \to 0$ then

$$\lim \frac{\partial}{\partial \varepsilon}\varphi(x^0 + \varepsilon\bar{x}) = 0. \tag{1.2}$$

This means that the half-line $x^0 + t\bar{x}$, where $t > 0$, touches the surface $\varphi = 0$ at the point x^0. since

$$\frac{\partial}{\partial \varepsilon}\varphi(x^0 + \varepsilon\bar{x}) = \frac{\partial \varphi}{\partial x_1}\bar{x}_1 + \cdots + \frac{\partial \varphi}{\partial x_n}\bar{x}_n,$$

where the partial derivatives are taken at the point $x = x^0$, the variation admissible to the restriction $\varphi = 0$, defined by linear inequality with constant coefficients. Since x^0 gives the extremum, the sets variations determined by the conditions (1.1) and (1.2) do not have common points. Conditions of nonintersecting recorded by applying determining their linear form, it is a necessary condition for an extremum.

It is particularly important for us to emphasize the following fact. The problem of finding the necessary conditions has the problem of finding conditions nonintersecting some sets defined by linear forms.

We also note that the sets defined by linear inequalities and equality are convex. In this case, these conditions are easily obtained from the following considerations: set defined by (1.2), should coincide with the set, defined by the equation

$$\frac{\partial f}{\partial x_1}\bar{x}_1 + \cdots + \frac{\partial f}{\partial x_n}\bar{x}_n = 0.$$

But if the two linear forms of the set of zeros coincide, their coefficients characterized by a constant factor; so there is λ, which

$$\frac{\partial f}{\partial x_1} = \lambda\frac{\partial \varphi}{\partial x_1},$$

$$\cdots \qquad \cdots$$

$$\frac{\partial f}{\partial x_n} = \lambda\frac{\partial \varphi}{\partial x_n}.$$

Now we proceed to obtain the necessary conditions of the second order. It is known, these conditions are the following.

The quadratic form satisfies

$$\sum_{i,k} \frac{\partial^2 H}{\partial x_i \partial x_k} \xi_i \xi_k \geqslant 0$$

for all those ξ which satisfy the equation

$$\frac{\partial \varphi}{\partial x_1}\xi_1 + \frac{\partial \varphi}{\partial x_2}\xi_2 + \cdots + \frac{\partial \varphi}{\partial x_n}\xi_n = 0.$$

Let us obtain following last condition by using geometrical reviews, fix the variation of $\bar{x}$ that satisfies the equation

$$\frac{\partial \varphi}{\partial x_1}\bar{x}_1 + \cdots + \frac{\partial \varphi}{\partial x_n}\bar{x}_n = 0.$$

Due to the necessary conditions of the first order, derivative of the functional f in the $\bar{x}$ direction is zero.

We call the second variation $\tilde{x}$ prohibited if the $\varepsilon > 0$ and $\to> 0$

$$\lim \frac{\partial^2}{\partial \varepsilon^2} f(x^0 + \varepsilon \bar{x} + \frac{\varepsilon^2}{2}\tilde{x}) < 0. \tag{1.3}$$

We are interested in the second derivative is as follows:

$$\frac{\partial f}{\partial \bar{x}_1}\tilde{x}_1 + \cdots + \frac{\partial f}{\partial \bar{x}_n}\tilde{x}_n + \sum_{i,k} \frac{\partial^2 f}{\partial x_i \partial x_k}\bar{x}_i \bar{x}_k.$$

Thus, the second prohibited defines some variation $\bar{x}$ linear inequality with a free term, depending on the variation of $\tilde{x}$. We call forth x permissible to restrict $\varphi = 0$ if

$$\lim \frac{\partial^2}{\partial \varepsilon^2} \varphi(x^0 + \varepsilon \bar{x} + \frac{\varepsilon^2}{2}\tilde{x}) = 0 \tag{1.4}$$

if $\varepsilon > 0$ and $\varepsilon \to 0$

This condition means that the parabola $x^0 + t\bar{x} + (t^2/2)$ tangent at x^0 surface $\varphi = 0$ up to small higher than second order in t. Equality (1.4) can be opened as follows:

$$\frac{\partial \varphi}{\partial x_1}\tilde{x}_1 + \cdots + \frac{\partial \varphi}{\partial x_n}\tilde{x}_n + \sum_{i,k} \frac{\partial^2 \varphi}{\partial x_i \partial x_k}\bar{x}_i \bar{x}_k = 0$$

Thus, the second admissible variation $\tilde{x}$ identifies some linear inequality with a free term, depending on the variation of $\tilde{x}$. Since x^0 gives the extremum, the plurality of second variations of $\tilde{x}$, defined (1.3) and (1.4) do not have common points. As well as before, the condition of their nonintersection recorded by applying and free members objectified.

Just as above, we call attention to the fact that the necessary condition of extreme acts as a condition of some nonintersection convex sets. In this case, the problem is solved as follows. Expression :

$$\frac{\partial f}{\partial x_1}\tilde{x}_1 + \cdots + \frac{\partial f}{\partial x_n}\tilde{x_n} + \sum_{i,k} \frac{\partial^2 f}{\partial x_i \partial x_k}\bar{x}_k\bar{x}_i \tag{1.5}$$

must be a negative constant for all $\tilde{x}$ that satisfy condition (1.4). And since $\nabla f = \lambda \nabla \varphi$, then, substituting $\lambda(\frac{\partial \varphi}{\partial x_i})$ (1.5) instead of $\frac{\partial f}{\partial x_i}$ get

$$0 \leqslant \delta = \lambda \frac{\partial \varphi}{\partial x_1}\tilde{x}_1 + \cdots + \lambda \frac{\partial \varphi}{\partial x_n}\tilde{x}_n + \sum_{i,k} \frac{\partial^2 f}{\partial x_i \partial x_k}\bar{x}_k\bar{x}_i \; .$$

but according to (1.4)

$$\frac{\partial \varphi}{\partial x_1}\tilde{x}_1 + \cdots + \frac{\partial \varphi}{\partial x_n}\tilde{x}_n = -\sum_{i,k} \frac{\partial^2 \varphi}{\partial x_i \partial x_k}\bar{x}_k\bar{x}_i \; .$$

Finally, we obtain

$$\sum_{i,k} \frac{\partial^2 f}{\partial x_i \partial x_k}\bar{x}_k\bar{x}_i - \lambda \sum_{i,k} \frac{\partial^2 \varphi}{\partial x_i \partial x_k}\bar{x}_k\bar{x}_i = \delta \geqslant 0 \; .$$

We arrive at the stated above known necessary condition extreme.

We now note that the two terms as first and second order, we got a result, substantially the same considerations, both conditions can be regarded as the Euler equations for the corresponding tasks. The method is to ensure that we identify some many curves along which explore the problem of the extremum, and enter a description of them, in which there is a plurality of predetermined using linear relations are convex. Necessary extremum condition then reduces to the condition of nonintersection convex sets. However, there are such problems, which arising sets are not convex, in this case this set divide them into convex set most convenient way.

1.1. The First Variation

In this section we now present the general scheme of the method, which we have illustrated with examples in the previous section.

Suppose that in a complete normed space W is given functional $F(W)$. It required to find the minimum of this functional for some conditions restricting the set of values of w. Let w^0 is the minimum point. We will explore the functionals and restrictions in a neighborhood of w^0. We assume here that is set a finite number of constraints such as inequality and equality constraints. Each inequality constraints given by the set, which is the closure of some open to the W of the set. restrictions equality type emit a closed set in W, and open part of which is empty.

We call $\overline{w}$ prohibited variation if there is a neighborhood $U_{\overline{w}}$ and $\varepsilon_0 > 0$ such that $F(w^0 + \varepsilon\overline{w}_1) < F(w^0)$ with at $0 < \varepsilon < \varepsilon_0$ and $\overline{w}_1 \in U_{\overline{w}}$. We assume that a set of prohibited variation w is not empty.

It can be seen from the definition of prohibited variations, in the case when the set of prohibited variations are not empty the it make an open cone with vertex at the origin.

We call w as the prohibited variation on some type of restriction inequality, if there is a neighborhood in the $U_{\overline{w}}$ and $\varepsilon_0 > 0$ such that $0 < \varepsilon < \varepsilon_0$ and $\overline{w}_1 \in U_{\overline{w}}$ point $w^0 + \varepsilon\overline{w}_1$ satisfy this constraint. We assume here that many variations allowed for each of the restrictions of inequality type is not empty. In this case, as well as for the prohibited variation, it is clear that the set of prohibited variations on this inequality constraint form an open cone with vertex at the origin. We call $\overline{w}$ admissible variation for equality constraints if, whatever the neighborhood $U_{\overline{w}}$ and whatever $\varepsilon_0 > 0$, there are always about $0 < \varepsilon < \varepsilon_0$ and the $\bar{w}_1 \in U_{\overline{w}}$ that the point $w^0 + \varepsilon\overline{w}_1$ satisfies the equality constraints. Easily It is seen that in the case where the set of admissible $\overline{\omega}$ is not empty, it is a closed cone with vertex at the origin.

We now denote cone of prohibited variations with Ω_0 , cone of variations which admissible for the i-th inequality constraints with Ω_i and finally, the cone of variations admissible for restrictions inequality type by Ω. Because of the design is evident that if the cones Ω_0, $\Omega_1, \ldots, \Omega_n$ and Ω are not empty intersection, the point w^0 could not be a point of minimum of the functional $F(w)$ under these restrictions.

Indeed, let $\overline{w}^0 \neq 0$, $\overline{w}^0 \in \Omega_0\Omega_1, \ldots, \Omega_n\Omega$. Since the intersection a finite number of neighborhoods of $\overline{w}^0$ is also a neighborhood of $\overline{w}^0$, there is a a neighborhood $U_{\overline{w}^0}$ and is $\varepsilon_0 > 0$, as soon as that $0 < \varepsilon < \varepsilon_0$ and we $\bar{w} \in U_{\overline{w}^0}$, then $F(w^0) > F(w^0 + \varepsilon\overline{w})$ and the point $w^0 + \varepsilon\overline{w}$ satisfies all inequality constraints. Since

$\overline{w}^0 \in \Omega$, i.e. Is prohibited for the equality restriction, then there is $0 < \varepsilon_1 < \varepsilon_0$ and the $\bar{w}_1 \in U_{\overline{w}^0}$, that the point $w^0 + \varepsilon_1 \overline{w}_1$ satisfies all the constraints of equality type. But, it is already mentioned, this is same point satisfies all the inequality constraints and the value it functional F at this point less then $F(w^0)$. So, the requirement that the intersection $\Omega_0 \Omega_1, \ldots, \Omega_n \Omega$. was empty, is a minimum condition. This condition is easily stored in the case, when all the cones are convex. Any convex set can be set using finite or infinite number of linear inequalities. It is therefore natural to formulate the condition of non-intersection of convex cones by means of the linear inequalities, which they are set. Here is the wording of this condition.

Theorem 1.1 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1968)) Let $\Omega_0, \Omega_1, \ldots, \Omega_n$ be an open convex cone with vertex at the origin and Ω be closed convex cone with vertex at the origin. Then, to the intersection of all of the cones was empty if and only if there exist linear functional $\omega_0, \omega_1, \ldots, \omega_n$, ω with such properties*

(1) $\omega_0 + \omega_1 + \cdots + \omega_n + \omega = 0,$ (1.6)

(2) *not all functionals are equal to zero,*

(3) $\omega_i(\Omega_i) \geqslant 0, \ i = 0, 1, \ldots, n, \omega(\Omega) \geqslant 0$

Equation (1.6) is called the Euler equation.

1.2. The Dual Cone

Definition 1.1 *Let Ω be a convex cone. The set of linear functional, non-negative on Ω is called a dual cone and denoted by Ω^* .*

We give a formulation of the theory concerning the relationship with cones. They conjugate.

Theorems 1.2 and 1.4 have not been previously known.

Theorem 1.2^o (Dubovitskii, A.Ya. and Milyutin, A.A. (1968)) *Let M be an open convex set, N be a convex set. If $M \bigcap N$ is empty there exists a linear functional λ separating the sets M and N, i.e. such that for any pair of elements $m \in M$ and $n \in N$, the inequality $\lambda(m) < \lambda(n)$.*

Theorem 1.2 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1968)) Let there are given an open convex cone $\Omega_1, \dots, \Omega_n$ and convex cone of Ω . For the intersection of these cones to be empty, it is necessary and sufficient that there exist linear functional $\omega_1, \dots, \omega_n, \omega$ where $\omega_i \in \Omega_i{}^*$, $\omega \in \Omega^*$ not all equal zero such that $\omega_1 + \omega_2 + \cdots + \omega_n + \omega = 0$*

Theorem 1.3 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1968)) Let Ω be an open convex cone with vertex the origin and L be a subspace of W. Let Ω' the intersection of Ω and L. Then if Ω' is not empty and any linear functional λ', which is defined on L and non-negative on Ω', can be extended to the whole space so that the extension will belong Ω^*.*

This theorem is equivalent to the following

Theorem 1.3′ (Dubovitskii, A.Ya. and Milyutin, A.A. (1968)) Let X and Y are complete normed space, A be linear operator mapping X into Y, Ω_y be an open convex cone in Y, Ω_x be the complete inverse image Cone Ω_y when displaying A. If Ω_x is not empty, then any linear functional $l(x) \in \Omega_x{}^*$ can be represented as $A^*\lambda[y(x)]$, where $\lambda(y) \in \Omega_y{}^*$ (Lemma Farkaşa - Minkovskoki).

Theorem 1.4 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1968)) Suppose there are two convex cone Ω_1 and Ω_2, and Ω_1 be an open cone. Then, if $(\Omega_1 \bigcap \Omega_2)$ is not empty, then $(\Omega_1 \bigcap \Omega_2)^* = \Omega_1{}^* + \Omega_2{}^*$.*

Theorem 1.5 *(Dmitruk, A.V. (1990)) Let X and Y be a complete normed space and A be a linear operator from X to Y. In the space of pairs $Z = (X, Y)$ consider the subspace $L : Y = AX$; then L^* consists of all the functional, which have the form $l(y - Ax)$, where l ba an arbitrary linear functional which is defined on L.*

Theorem 1.6 *(Dmitruk, A.V. (1990)) Let L_1 and L_2 are a subspace of W. If $L_1 + L_2$ is a subspace (closed linear manifold), then $(L_1 \bigcap L_2)^* = L_1{}^* + L_2{}^*$.*

1.3. Linear Convex Functionals

In this section we consider the so-called linear convex functional, which will play a significant role in the future the study of certain cones.

Definition 1.2 *A functional $f(x)$ is called linear convex functional which is defined on the elements x of a complete normed space X, with the following three properties:*

(1) $f(\alpha x) = \alpha f(x);$ *if* $\alpha > 0$

(2) $f(x_1 + x_2) \leqslant f(x_1) + f(x_2);$

(3) $|f(x)| \leqslant c||x||$, *for constant* $c > 0$.

Note that the properties (1) and (3) follow the continuity of the functional $f(x)$.

Here are some examples of linear convex functional.

1. Let $f(x) = l(x)$, where $l(x)$ is a linear functional, then, obviously, $f(x)$ is linear convex functional.

2. Also $f(x) = ||x||$ is linear convex functional.

3. $f(x) = ||Ax||$ where A is bounded linear operator,$f(x)$ is linear convex functional We formulate some properties of linear convex functional, that we will use later on:

(1) Let $f(x)$ are linear convex functional; then $\alpha f(x)$ where (α is a negative number) is obviously a linear convex functional,

(2) Let $f_1(x)$ and $f_2(x)$ are linear convex functional; then $f(x) = f_1(x) + f_2(x)$ is also a linear convex functional,

(3) Let there be an arbitrary set $\{f_\alpha\}$ is uniformly bounded linear convex functional; then $f(x) = \sup_\alpha f_\alpha$ also linear convex functional.

We define the important concept of supporting functional . Let $f(x)$ be a linear convex functional. A linear functional $\mu(x)$ is called a supporting to the functional $f(x)$, if $f(x) \geqslant \mu(x)$ for any x. We have the following

Definition 1.3 *Let X be a locally convex space, and $C \subset X$ be a convex set and $f : C \to \mathbb{R}$, then the continuous linear functional $\mu : X \to \mathbb{R}$ is a supporting functional of f if $f(x) \geqslant \mu(x)$ for every $x \in C$.*

Theorem 1.7 *(Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004)) For any x_0 there exists a supporting functional $\mu(x)$ such that $\mu(x_0) = f(x_0)$.*

Let us point out some obvious properties of the set of supporting functionals, arising from the definition of the supporting functional and Theorem (1.7):

(1) Let $f(x)$ are linear convex functional; then the set supporting functional is bounded;

(2) A set of supporting functionals is a convex set;

(3) A set of supporting functionals is a closed set;

(4) Linear convex functional $f(x)$ is uniquely determined by the set their supporting functionals, and $f(x) = \max \mu(x)$ for all the supporting functionals μ.

Here are some examples of supporting functional.

(1) Let $f(x) = l(x)$, where $l(x)$ is a linear functional. To that there is only one functional supporting functional, matching with himself. In fact, the inequality $l(x) = \mu(x)$ follows equality $l(x) = \mu(x)$.

(2) Let $f(x) = ||x||$. The set of supporting functional consists such functional which norm of this functional no greater then one. The rate does not exceed unity. Therefore, in this case, set of supporting makes unique sphere .

The following theorem and suggestions allow us to find supporting functional.

Theorem 1.8 *(Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004)) Let $\varphi(x)$ be a linear convex functional, L be subspace of X , $\mu_L(x)$ be a linear functional define on L and supporting for $\varphi(x)$ on L then there exist functional $\mu(x)$ defined on X supporting functional for $\varphi(x)$ and $\mu(x) = \mu_L(x) \quad if \quad x \in L$.*

Theorem 1.9 *(Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004)) Let $f(x) = f_1(x) + f_2(x)$ where $f_1(x)$ and $f_2(x)$ are linear convex functional $\mu(x)$ is supporting functional for $f(x)$ iff $\mu(x) = \mu_1(x) + \mu_2(x)$ where $\mu_1(x)$ is supporting functional for $f_1(x)$ and $\mu_2(x)$ is supporting functional for $f_2(x)$.*

Theorem 1.10 *(Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004)) Let $f_1(x)$ and $f_2(x)$ are linear convex functional and $f(x) = \max(f_1(x), f_2(x))$. In order to $\mu(x)$ was a supporting to $f(x)$ functional, it is necessary and sufficient that $\mu(x) = \alpha\mu_1(x) + \beta\mu_2(x)$ where $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$; $\mu_1(x)$ and $\mu_2(x)$ supporting functional $f_1(x)$ and $f_2(x)$, respectively.*

Theorem 1.11 *(Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004)) Let $f_1(x)$ and $f_2(x)$ are linear convex functional. Let $|f_1(x) - f_2(x)| \leqslant c\|x\|$. Then, whatever the functional $\mu_1(x)$, a supporting to the $f_1(x)$, there is a supporting functional $\mu_2(x)$, for the $f_2(x)$ such that $\|\mu_1(x) - \mu_2(x)\| \leqslant c$, and Conversely.*

Consider the homogeneous convex functional of two real variables $F(\xi, \eta)$. Such a functional can be considered as an example of linear convex functional defined on the plane. As we have shown above, $F(\xi, \eta) = \max(\mu'(\xi, \eta))$ around the μ', where μ' is linear functional two variables satisfies the inequality $\mu'(\xi, \eta) \leqslant F(\xi, \eta)$. Assume that $F(\xi, \eta)$ which has a form $\mu'(\xi, \eta) = a\xi + b\eta$, be a supporting functional for $F(\xi, \eta)$, a and b are non-negative. Let $f_1(x)$ and $f_2(x)$ are linear convex functional. Then the functional $\varphi(x) = F(f_1(x)\,, f_2(x))$ linear convex functional. Let us take this supporting functional in condition $\varphi(x) = F(f_1(x)\,, f_2(x))$.

Theorem 1.12 *(Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004)) A linear functional $\mu(x)$ is the supporting functional for $\varphi(x)$ if and only if there exists a linear functional two variables$\mu'(\xi, \eta) = a\xi + b\eta$ that is the supporting functional for $F(\xi, \eta)$ and $\mu(x)$ is a supporting functional to the $\varphi_{\mu'}(x) = af_1(x) + bf_2(x)$.*

Note. We note that the theorem holds for the case of convex functionals $f(\xi_1, \ldots, \xi_n)$ of any finite number of variables.

Let A be a bounded linear operator mapping X into Y. Suppose that in Y given a linear convex functional $f(y)$. We consider the functional $\varphi(x) = f(Ax)$. It is easy to see that $\varphi(x)$ is a linear convex functional. By using functional, supporting functional for the $f(y)$, we find functional, supporting for $\varphi(x)$.

Theorem 1.13 *(Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P. (2004)) For a linear functional $\mu(x)$ be supporting for $\varphi(x)$ is necessary and sufficient for it to be represented in the form $\mu(x) = v(Ax)$, where $v(y)$ is a supporting functional for $f(y)$.*

In conclusion, it seems good for us to provide a brief overview of the results of this section. Thus, as above, $f_r(x)$, $r = 1, 2, \cdots$ are Linear convex functionals, $\mu_r(x)$, $r = 1, 2, \cdots$ are supporting functionals them accordingly linear functions. Next,$F(\xi_1, \dots, \xi_n)$ is convex functional of n variables, $\mu'(\xi_1, \dots, \xi_n)$ be the supporting for it a linear form.

Theorems (1.7) and (1.8) are devoted to the problem of the existence of supporting functionals; they imply, in particular set of supporting functionals uniquely identifies linear convex functional. Theorems (1.9) and (1.12) establish a connection between the operations of the linear convex functionals on sets and supporting them functional. Through these theorems in some cases easily find a lot of supporting functionals for the linear convex functional. The results these theorems can be conveniently represented in the form of regulations in Table. (1.1).

Linear convex functional	Supporting functional
$f(x) = L(x)$	$\mu(x) = L(x)$
$\sum_1^n \alpha_i f_i \quad , \quad \alpha_i \geqslant 0$	$\sum_1^n \alpha_i \mu_i$
$max(f_1, \dots, f_n)$	$\alpha_1\mu_1 + \cdots + \alpha_n\mu_n \; , \alpha_i \geqslant 0 \quad , \sum_1^n \alpha_i = 1$
$F(f_1, \dots, f_n)$	$\mu'(\mu_1, \dots, \mu_n)$
$f(Ax)$	$\mu(Ax)$

Table 1.1: Relation between supporting functionals and linear convex functional

The functional $F(\xi_1, \dots, \xi_n)$ such that any supporting coefficients linear form μ' Non-negative.

Let $f(x) = |L(x)|$ where L is linear convex functional

It is clear

$f(x) = max\{L(x), -L(x)\}$

then by Table 1.1

$\mu(x) = (\alpha + \beta)L(x), \qquad \alpha + \beta = 1$

$\mu(x) = \alpha\mu_0(x) + \beta\mu_2(x)$

$= \alpha L(x) - \beta L(x)$

$= (\alpha - \beta)L(x)$

1.4. Extremum Problems With The Constraints

Example 1.1 *Let $f(x) = |l(x)|$, where $l(x)$ is a linear functional. Obviously,$f(x) = max(l(x); -l(x))$. According to the theorem (1.10), $\mu(x) = (\alpha - \beta)l(x)$,where $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$. Putting $\alpha - \beta = \gamma$, we obtain $\mu(x) = \gamma l(x)$, where $|\gamma| \leqslant 1$.*

Example 1.2 *Let $f(x) = max(l_1(x), \dots, l_n(x))$. According to the theorem (1.10) $\mu(x) = \alpha_1 l_1(x) + \cdots + \alpha_n l_n(x)$, where $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$.*

Example 1.3 *Let $f(x) = max(|l_1(x)|, ..., |l_n(x)|)$. By using the results of example(1.1) and theorem (1.9), we have $\mu(x) = \sum \alpha_i \gamma_i l_i(x)$, $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$, $|\gamma_i| \leqslant 1$, Assuming $(\alpha_i \gamma_i = \gamma'_i)$ then we have $\mu(x) = \sum \gamma'_i l_i(x)$ and $\sum |\gamma'_i| \leqslant 1$.*

Example 1.4 *Let $f(x) = |l_1(x)| + \cdots + |l_n(x)|$. Let us use theorem (1.10), $F(\xi_1, \dots, \xi_n) = \xi_1 + \cdots + \xi_n$ where $\mu'(\xi_1, \dots, \xi_n) = \xi_1 + \cdots + \xi_n$ by theorem (1.10) $\mu(x) = \mu_1(x) + \cdots + \mu_n(x)$ where $\mu_i(x)$ is a supporting functional to $|l_i(x)|$. Using the result of Example 4.1, we finally obtain*

$$\mu(x) = \sum \gamma_i l_i(x), \qquad |\gamma_i| \leqslant 1 .$$

Example 1.5 *Consider the space of continuous functionals $x(t)$, defined on the interval (0,1). Let $f(x) = \max x(t)$.*
It is easy to verify that $f(x)$ are linear convex functional. Let us find general form of the supporting functional. It is known that any linear functional in the space of continuous functionals can be represented as $\int x(t)dv$, where v be some completely additive measure. We describe those v, for which us take $\int x(t)dv = \mu(x)$. Let the measure v_0 belongs to this class. Let n be a positive integer. Let $t_n = k/n$, where $k = 1, \dots, n$. Consider the $(n+1)$ dimensional subspace of R_{n+1} continuous space functionals, which we define as follows: $x(t) \in R_{n+1}$ and then only if $x(t)$ is linear in every interval between two adjacent the points. Obviously, if $x(t) \in R_{n+1}$ then

$$f(x) = \max_{k=0,1,\dots,n} x(t_k).$$

Consider $\mu(x) = \int x(t)dv_0$ *on the elements of the subspace* R_{n+1}. *If* $x(t_k)$ *is a linear functional, then, according to the result of example 4.2,*

$$\int x(t)dv_0 = \sum_k \alpha_k x(t_k),$$

where $\alpha_k \geqslant 0$ *and* $\sum \alpha_k = 1$ *for all* $x(t) \in R_{n+1}$. *Letting n tend to infinity, We obtain that* $v_0(t)$ *be weak limit of a sequence non-negative measures, a complete change of which is equal to one. It is known that the measure* v_0 *should also have these two properties. Thus we have shown that if* $\int x(t)dv_0 = \mu(x)$, *then* $dv_0 \geqslant 0$, $\int dv_0 = 1$, *It is also easy to see that if the measure v has two properties, the* $\int x(t)dv = \mu(x)$.
Indeed,

$$\int x(t)dv \leqslant \max x(t) \int dv = \max x(t) = f(x).$$

For the generalization let us take arbitrary set. Now let M be a closed bounded set in a finite dimensional space $(\xi_1, \ldots, \xi_n)$. Consider the space of continuous functionals $x(\xi_1, \ldots, \xi_n)$, defined on the set M. Let take

$$f(x) = \max_{\xi} x(\xi_1, \ldots, \xi_n)$$

In this case, we obtain $\mu(x) = \int x(\xi)dv$ where v is a non-negative measure concentrated on M and complete v is the change unit.
Now let us consider the following question: Let $x_0(t)$ is a continuous functional on the interval [0,1] , and take $f(x) = \max_t x(t)$. Let us discuss those supporting functionals $\mu(x)$, which satisfy the equation $\mu(x_0) = f(x_0)$. As described above,

$$\mu(x) = \int_0^1 x(t)dv , \qquad where,\ dv \geqslant 0, \int_0^1 dv = 1$$

If the functional $\mu(x)$ satisfies $\mu(x_0) = f(x_0)$, it means that

$$\int_0^1 x_0(t)dv = \max_t x_0(t)$$

Denote with the set M_0 such values of t , for which $x_0(t) = f(x_o)$. It is clear, the measure v should focus on M_0. It is easy to see that this circumstance completely characterizes the class of supporting functionals, which at $x_0(t)$ coincide with $f(x_0)$.
In what follows we will be useful following reformulation subordinate functional properties this type:
Linear functional $\mu(x)$ is a supporting to $f(x)$ and $\mu(x_0) = f(x_0)$, if and only if following conditions satisfies:

(1) $\mu(x)$ is a non-negative functional; it means that $\mu(x) \geqslant 0$ if $x(t) \geqslant 0$ $(dv \geqslant 0)$ for all t;

(2) $\mu(1) = 1$;

(3) For every functional $x(t)$, which vanishes on the set M_0, $\mu(x) = 0$.

Example 1.6 *Consider again the space of continuous functionals, defined on the interval [0,1]. Let M is a closed subset of [0,1]. Let*

$$f(x) = \max_{t \in M} x(t)$$

In order to find a general form of the supporting functional, it is convenient to use the following method:

The space of continuous functionals, defined on [0,1] denote the $C_{0,1}$. *The space of continuous functionals,defined on M, denoted by* C_M. *Consider the linear mapping A from the space* $C_{0,1}$ *to the space* C_M. *when* $\underset{t \in M}{x_1(t)} = \underset{t \in M}{x(t)}$

In the space C_M *we consider linear convex functional*

$$f_1(x_1) = \max_t \underset{t \in M}{x_1(t)}$$

It is clear. $f(x) = f_1(Ax)$.

From the theorem 1.11 and the result of the previous example, we have $\mu(x) = \int x(t)dv$ *, where v be a non-negative measure concentrated on M full change equal to one.*

Example 1.7 *Consider the space of bounded measurable functionals* $u(t)$ *defined on the interval [0,1]. Let*

$$\|u(t)\| = \text{vrai}\max_t |u(t)| \, , \qquad F(u) = \text{vrai}\max_t u(t).$$

It is easy to see that $F(u)$ *are linear convex functional. characterize set of functional, supporting to the functional* $F(u)$. *Same as in Example 1.6, we prove that for a linear functional* $\mu(u)$ *was a supporting to* $F(u)$, *it is necessary and sufficient that* $\mu(u)$ *has the following two properties:*

(1) $\mu(u) \geqslant 0$, *if* $u \geqslant 0$;

(2) $\mu(u \equiv 1) = 1$.

Let us prove the necessity. Let $\mu(u)$ *be a supporting functional,* $u_0(t) \geqslant 0$*. Suppose that*$\mu(u) < 0$*. Then*

$$\mu(-u_0) > 0 \geqslant \operatorname{vrai}\max_t(-u_0(t)),$$

It is impossible. Thus, the need for property (1) proved. Further,obviously, $\mu(u \equiv 1) \leqslant 1$, $\mu(u \equiv -1) = -\mu(u \equiv 1) \leqslant -1$ *. Both inequalities derive from the fact that* $\mu(u) \leqslant F(u)$ *for all u. The inequalities seen that* $\mu(u \equiv 1) = 1$ *. The need for the property (2) is proved. Let us prove the sufficiency. Assume that the functional* $\mu(u)$ *has properties (1) and (2). Let* $u_0(t)$ *is a functional and* $c = F(u_0)$*. Then* $c = \mu(u \equiv c) = \mu(u_0) + \mu(c - u_0) \geqslant \mu(u_0)$ *as* $(c - u_0) \geqslant 0$ *. The sufficiency is proved.*

Let $u_0(t)$ be some functional. We characterize the set of all the supporting functionals that satisfy the equation $F(u_0) = \mu(u_0)$. in example 1.6 the case of continuous functionals, we showed Such functional concentrated on the set of points t, where $x(t) = f(x)$. For measurable functionals set on which functional reaches its maximum value, it may be empty. Therefore characteristic of supporting functionals satisfying equality $\mu(u_0) = F(u_0)$, is somewhat more complicated, essentially reflecting the same property. Let $\delta > 0$. We define M_δ the set as follows way $t \in M_\delta$: if $u_0(t) > F(u_0)$ - δ. Obviously, M_δ not empty and has positive measure. We show that in order to supporting functional $\mu(u)$ satisfies the equation $\mu(u_0) = F(u_0)$ it is necessary and sufficient so that, $\mu(u) = 0$ for any functional $u(t)$, vanishing on some M_δ.

Let us prove the necessity. Assume that $u_1(t)$ such that $u_1(\mu_\delta) \equiv 0$ for some $\delta > 0$ and $\mu(u_1) \neq 0$. We can always assume that $\mu(u_1) > 0$ and that $\|u_1(t)\| = 1$. Consider $u(t) = u_0(t) + \delta u_1(t)$. It is easy to see that $F(u) = F(u_0)$. On the other hand, $\mu(u) > \mu(u_0) = F(u_0) = F(u)$. The last inequality contradicts the assumption that $\mu(u)$ is a supporting functional. Therefore, the opinion is proved.

Let us prove sufficiency. Let $\mu(u)$ be a supporting functional equal zero for any functional $u(t)$, vanishing on some M_δ. We define functional $u_1(t)$, fixing $\delta > 0$:

$$u_1(t) = \begin{cases} u_0(t) & \text{if } t \in M_\delta; \\ F(u_0) & \text{for others } t\ . \end{cases}$$

It is easy to see that $\mu(u_1) = \mu(u_0) \geqslant F(u_0) - \delta$. Since δ is an arbitrary number, $\mu(u_0) = F(u_0)$. The sufficiency is proved.

Example 1.8 *Consider the space of integrable functionals square on the interval [0,1]. Let* $x^+(t) = \max\,(x(t), 0)$. *Let*

$$f(x) = \sqrt{\int_0^1 [x^+(t)]^2 dt}.$$

Let us prove that $f(x)$ *is linear convex functional. In fact, firstly,* $f(\alpha x) = \alpha f(x)$ *at* $\alpha \geqslant 0$ *; secondly, it is obvious from the boundedness of the functional* $f(x)(x^+(t) \leqslant |x(t)|\,)$. *Further,*

$$f(x_1 + x_2) = \sqrt{\int_0^1 [(x_1 + x_2)^+]^2 dt} \leqslant \sqrt{\int_0^1 ({x_1}^+ + {x_2}^+)^2 dt} \leqslant$$

$$\leqslant \sqrt{\int_0^1 ({x_1}^+)^2 dt} + \sqrt{\int_0^1 ({x_2}^+)^2 dt} = f(x_1) + f(x_2)$$

Thus, all the three properties ε_0 determine linear convex functional are satisfied. It is known that the general form of linear functional $l(x)$ in the space $L_2(x)$ have $l(x) = \int \psi(t)x(t)dt$, where $\psi(t) \in L_2$. We describe the class of $\psi(t)$, for which $\int \psi(t)x(t)dt = \mu(x)$. Let $\psi_0(t)$ belongs to this class. Let n positive integer. Let $t_k = k/n, k = 0, 1, \ldots, n$. We now consider R-dimensional subspace R_n space L_2 of functionals equal to a constant on each interval between adjacent points t_k. If $x(t) \in R_n$ and

$$f(x) = \frac{1}{\sqrt{n}} \sqrt{\sum_{k=1}^{n} [x_k^+]^2}\,,$$

wherein x_h - values $x(t)$ at the $k - M$ interval t_k. Consider

$$F(\xi_1, \ldots, \xi_n) = \max_{a_i}(\alpha_1\xi_1, \ldots, \alpha_n\xi_n), \quad a_i \geqslant 0 \quad \sum {a_i}^2 \leqslant 1\,.$$

Obviously, $f(x) = (1/\sqrt{n})F({x_1}^+, ..., {x_n}^+)$ $({x_i}^+ \geqslant 0)$ for $x(t) \in R_n$.

By using Theorem 1.10, we see that on R_n

$$\int \psi_0(t)x(t)dt = \frac{1}{\sqrt{n}} \sum \alpha_i \mu_i(x)$$

where $\alpha_i \geqslant 0$, $\sum {\alpha_i}^2 \leqslant 1$, $\mu_i(x)$, is the supporting to the linear convex functional ${x_1}^+$. Since ${x_1}^+ = \max\,[x_i, 0]$ then $\mu_i(x) = \alpha_i x_i$ where $0 \leqslant \alpha_i \leqslant 1$. In this way,

$$\int \psi_0(t)x(t)dt = \frac{1}{\sqrt{n}} \sum \alpha_i \alpha_i x_i$$

Assuming $\alpha_i \alpha_i \sqrt{n} = \psi_i$, we get

$$\int \psi_0(t)\, x(t)\, dt = \int \psi(t)\, x(t)\, dt$$

where $\psi(t), x(t) \in R_n$, $\psi(t) \geqslant 0$, $\int [\psi(t)]^2 dt \leqslant 1$. Letting n tend to infinity, we obtain $\psi_0(t)$ is a "weak" limit of a sequence negative functionals, the norm which is not exceeding one. It is known that $\psi_0(t)$ must also be non-negative and does not exceed the up to norm one. Thus if, $\int \psi_0(t)\, x(t)\, dt = \mu(x)$, then $\psi_0(t) \geqslant 0$ and $\int [\psi_0(t)]^2 dt \leqslant 1$. It is easy to see that if the functional $\psi(t)$ has such properties, then $\int \psi(t) x(t) dt = \mu(x)$. Indeed, in this case

$$\int \psi(t)\, x(t)\, dt \leqslant \int \psi(t)\, x^+(t)\, dt \leqslant \sqrt{\int [\psi(t)]^2 dt} \times$$

$$\times \sqrt{\int [x^+(t)]^2 dt} \leqslant \sqrt{\int [x^+(t)]^2 dt} = f(x)$$

Consider the case of an arbitrary set. Now let M - some measurable subset of the interval [0,1] of positive measure. Consider space $L_{(2,M)}$ functionals defined on M and such that

$$\int_M x^2(t) dt < \infty$$

In this space we consider linear convex functional

$$f(x) = \sqrt{\int_M [x^+(t)]^2 dt}$$

Almost literally repeating the preceding discussion, we conclude that

$$\mu(x) = \int_M \psi(t)\, x(t)\, dt,\ \psi(t) \in L_{2,M}$$

$$\psi(t) \geqslant 0\ , \quad \int_M \psi^2(t)\, x(t)\, dt \leqslant$$

Example 1.9 *Let M be a measurable subset of [0,1] with positive measure. In the space* $L_{2,[0,1]}$, *consider linear convex functional*

$$f(x) = \sqrt{\int_M [x^+(t)]^2 dt}.$$

By using exactly the same way as we did in the analysis Example 4.6, we obtain

$$\mu(x) = \int_M \psi(t)\, x(t)\, dt \quad \text{where} \quad \psi(t) \geqslant 0 \quad \text{and} \quad \int_M \psi^2(t) dt \leqslant 1$$

Example 1.10 *We now consider the space of continuous functionals. In this space we define a linear convex functional*

$$f(x) = \sqrt{\int_0^1 [x^+(t)]^2 dt}.$$

It is easy to see that this functional linear convex. Check this case subject only to the boundedness of functional, but it is easily obtained, as the norm in C restricted the top norm in L_2. In the example 1.8 we considered this functional in the space L_2, and there found a common view supporting functionals. Here we consider the functional in a narrower space and, therefore, we have a greater margin of linear functionals. The question arises whether the new will not appear to have found earlier (Example 1.8) supporting functional. We show that, as in the example 1.8, $\mu(x) = \int \psi\, x(t)\, dt \quad \psi(t) \geqslant 0$ and $\int \psi^2 \leqslant 1$. For this we consider the identity embedding of C in the space of L_2 . Obviously,that the investment can be regarded as a linear operator A mapping space $C_{[0,1]}$ in the space $L_{2,[0,1]}$. Then, applying the rule 1.5, we obtain the desired result. Thus, when "narrowing" of the space no new supporting functionals.

Example 1.11 *Let*

$$f(x) = \int_0^1 x^+(t)\, dt$$

It is convenient to consider this functional in the space L_1. We would consider this example as well, as we see an example of 1.8, however, here we use a simple and clear method. Infact, $x^+(t) = \frac{1}{2}x(t) + \frac{1}{2}|x(t)|$. In this way,

$$f(x) = \frac{1}{2}\int_0^1 x(t)\, dt + \frac{1}{2}\int_0^1 |x(t)|\, dt$$

By using theorem 1.8, we obtain $\mu(x) = \frac{1}{2}\mu_1(x) + \frac{1}{2}\mu_2(x)$. Here $\mu_1(x)$ is supporting to the $\int_0^1 x(t)dt$ and $\mu_2(x)$ is supporting to the $\int_0^1 |x(t)|\, dt$. Infact, $\int_0^1 x(t)dt$ is a linear functional, and $\int_0^1 |x(t)|\, dt$ is taken as a rule

$$\mu_1(x) = \int_0^1 x(t)\, dt, \quad \mu_2(x) = \int_0^1 \beta(t)\, x(t)\, dt,$$

where $|\beta(t) \leqslant 1$

Thus,

$$\mu(x) = \int_0^1 \frac{1+\beta(t)}{2}\, x(t)\, dt$$

Assuming that $\frac{1+\beta(t)}{2} = \theta(t)$ *, we obtain* $\mu(x) = \int_0^1 \theta(t)\, x(t)\, dt$ *, where* $0 \leqslant \theta(t) \leqslant 1$

$$f(x) = \int_M x^+(t)\, dt;$$

where M is a measurable subset of positive measure of [0,1] and $x(t) \in L_{1,[0,1]}$*. Arguing as in the analysis of example 1.6, we obtain*

$$\mu(x) = \int_M \theta(t)\, x(t)\, dt$$

where $0 \leqslant \underset{t \in M}{\theta(t)} \leqslant 1$

Let $a(t)$ *is some non-negative bounded measurable functional defined on the interval [0,1]. In the space* L_1 *consider functional*

$$f_1(x) = \int_0^1 \alpha(t)\, x^+(t)\, dt$$

We show that $f_1(x)$ *is linear convex functional. For this we consider bounded linear operator A mapping the space* L_1 *to itself as follows:*

$Ax = a(t)\, x(t)$*. Then,* $f_1(x) = f(Ax)$ *, where* $f(x)$ *is the functionality discussed above in this example. Since* $f(x)$ *is linear convex functional, then* $f_1(x)$ *is linear convex functionality for* $a(t)x^+(f) = [a(t)x(t)]^+$*. Under Rule 4.5, and the result Example 1.11,*

$$\mu_1(x) = \mu(Ax) = \int_0^1 \theta(t)a(t)x(t)dt$$

where $0 \leqslant \theta(t) \leqslant 1$

CHAPTER 2

SUPPORTING FUNCTIONALS AND LINEAR CONVEX FUNCTIONAL

2.1. Relation Between Supporting Functionals And Linear Convex Functional

1. We give a theorem by which the connection is established between supporting functionals to linear convex functional and functionals included in the dual cone, in the case where the original cone given by the linear convex functional. Let $f(x)$ be a linear convex functional and $x \in \Omega_x$ if and only when $f(x) < 0$. Let us assume that $f(x)$ is such that the cone Ω_x is not empty. Then it is easy to see that Ω_x is an open convex cone. In the fact, the continuity of $f(x)$ it follows that Ω_x is open set and homogeneity of $f(x)$ which Ω_x is cone. It remains to prove convexity. Let $x_1 \in \Omega$ and $x_2 \in \Omega_x$, then $f(x_1 + x_2) \leqslant f(x_1) + f(x_2) < 0$, and hence, $x_1 + x_2 \in \Omega$. Convexity proved. Let us find dual cone Ω_x.

Theorem 2.1 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1971)) A necessary and sufficient condition for a linear functional $l(x) \in \Omega_x$, is that the equality $l(x) = -\alpha\mu(x)$, hold where $\alpha \geqslant 0$, $\mu(x)$ is a supporting functional to the $f(x)$.*

2. Suppose now that $f_1(x)$ and $f_2(x)$ are linear convex functionals. Consider cone $\Omega_x = f_1(x) - f_2(x) > 0$. This cone usually is not convex, and its complement also is generally not convex. So, in order to write the Euler equation for that problem where it occurs cone of this type, we need classifying given cone to the numbers of convex of convex cones.

Here is an example, explaining said. In four-dimensional space $(\xi_1, \xi_2, \xi_3, \xi_4)$ consider the cone $\sqrt{{\xi_1}^2 + {\xi_2}^2} > \sqrt{{\xi_3}^2 + {\xi_4}^2}$. It is easy to see that this cone is not convex; it is sufficient to consider the cross section of the cone plane $\xi_2 = 0$ and $\xi_4 = 1$. We obtain the inequality ${\xi_1}^2 + {\xi_3}^2 \geqslant 1$ As we know, this set is not convex. Additional cone It derived from given by orthogonal transformation and, therefore also is not convex.

Let Ω_x be a cone (Ω_x is not empty), determined using the inequality $f_1(x) - f_2(x) > 0$, where $f_1(x)$ and $f_2(x)$ are linear convex functional. Let $\mu_1(x)$ is linear supporting functional to $f_1(x)$ and no supporting to $f_2(x)$ and $\varphi_{\mu_1}(x) = f_2(x) - \mu_1(x)$. It is clear , $\varphi_{\mu_1}(x)$ is linear convex functional. We denote by $\Omega_{\mu,x}$ convex open cone $\varphi_{\mu_1}(x) < 0$ (since $\mu_1(x)$ is not a supporting to $f_2(x)$, then the cone $\Omega_{\mu,x}$ does not empty). According to Theorem 2.1 and Theorem 1.8, the cone conjunct to $\Omega_{\mu,x}$ consent of the functionals type $-\alpha(\mu_2(x) - \mu_1(x))$, where $\alpha \geqslant 0$ $\mu_2(x)$ supporting to $f_2(x)$.

We have the following theorem:

Theorem 2.2 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1971))*

$$\Omega_x = \bigcup_{\mu_1} \Omega_{\mu_1,\, x,} \tag{2.1}$$

It means Ω_x is union of convex open sets $\Omega_{\mu_1,\, x}$

Proof : Let $x_0 \in \Omega_x$. This means that $f_1(x_0) > f_2(x_0)$. By Theorem 1.7, there exists a linear functional $\mu_1(x)$, a supporting to the $f_1(x)$ f such that $\mu_1(x_0) = f_1(x_0)$, therefore, $\mu_1(x_0) - f_2(x_0) > 0$ and $x \in \Omega_{\mu_1,\, x,}$ and then we get (2.1).

On the other hand, let

$$x_0 \in \bigcup_{\mu} \Omega_{\mu_1,\, x} \; ;$$

Then there is a linear functional ${\mu_1}^0(x)$, the supporting to the $f_1(x)$, that ${\mu_1}^0(x_0) > f_2(x_0)$, but then $f_1(x_0) > f_2(x_0)$, therefore,; $x_0 \in \Omega_x$. Therefore way again obtain theorem (2.1). QED. □

We can summarize all these result as a table

Convex Cone Ω_x	Dual Cone
2.1 $f(x) < 0$	$-\alpha\mu(x),\ \alpha \geqslant 0$
2.2 $f_1(x) - f_2(x) > 0$ stratified on the convex , $\mu_1(x) - f_2(x) > 0$,	$\alpha(\mu_1(x) - \mu_2(x)),\ \alpha \geqslant 0$

Table 2.1: Relation between convex cone and dual cone

In table. (2.1) $f(x)$ is a linear convex functional $\mu(x)$ is the linear supporting functional to $f(x)$, Ω_x is assumed to be non empty, $\mu_1(x)$ is not supporting to $f_2(x)$.

Example 2.1 *Let us take a cone $\Omega_x = l(x) < 0$. Since $l(x)$ is linear convex functional, according to the rule (2.1) and example (1.2), $\Omega_x{}^*$ consists of functionals which has a form $-\alpha l(x)$, where $\alpha \geqslant 0$.*

Example 2.2 *Let Ω_x is given by the inequalities $l_1(x) < 0, \ldots, l_n(x) < 0$. Assume that the Ω_x is not empty. Let us take $f(x) = \max[l_1(x), \ldots, l_n(x)]$. Then Ω_x is given by the inequality $f(x) < 0$. By using the rule (2.1) and example (1.2), we find that the general form of a functional $\Omega_x{}^*$ is $\sum \alpha_i l_i(x)$, $\alpha_i \geqslant 0$*

Example 2.3 *Let $l_1(x)$ and $l_2(x)$ are two linear independent functional. Consider the Ω_x, which is given by the inequality $|l_1(x)| - |l_2(x)| > 0$. Obviously cone Ω_x is not empty ($l_1(x), l_2(x)$ are linear independent) and is not convex. Let us separate it into convex cones in accordance with rule (2.2). In this case a linear functional $\mu_1(x)$, a supporting to the $|l_1(x)|$ has the form $\mu_1(x) = \gamma_1 l_1(x)$, where $|\gamma_1| \leqslant 1$. In order to $\mu_1(x)$ not to be a supporting to the $|l_2(x)|$, necessary and sufficient is that $\gamma_1 \neq 0$. So, Ω_x the cone splits into convex cones Ω_γ which is given by the inequality $\gamma l_1(x) - |l_2(x)| > 0$, $|\gamma| \leqslant 1$, $\gamma \neq 0$. It is easy to see that if γ_1 and γ_2 are the same sign and $|\gamma_1| > |\gamma_2|$ then $\Omega_{\gamma_1} \supseteq \Omega_{\gamma_2}$ In fact, the inequality $\gamma_2 l_1(x) > |l_2(x)|$ implies the inequality $\gamma_1 l_1(x) > |l_2(x)|$. Thus, the cone Ω_x can be represented as the union two open convex $\Omega_{\gamma,x}$, if $\gamma = 1$ and $\gamma = -1$ It is easy to see that these cones have not intersection. We now write, according to the rule (2.1), the general form of linear functional $\Omega_{1,x}{}^*$ and $\Omega_{-1,x}{}^*$. According to the rule (2.1), we have*

$$\begin{cases} \omega_1(x) = \alpha(l_1(x) - \beta l_2(x)), & |\beta| \leqslant 1, \quad \alpha \geqslant 0; \\ \omega_{-1}(x) = \alpha\{[-l_1(x)] - \beta l_2(x)\}, & |\beta| \leqslant 1, \quad \alpha \geqslant 0. \end{cases}$$

Example 2.4 *Let $l(x)$ to be a linear functional. Let $f(x) = l(x) - \|x\|$ and consider the cone Ω_x , which is given by the inequality $l(x) \geqslant \|x\|$. We assume that the Ω_x is not empty. In this case, Ω_x is an open cone. Next, we will try to show that the Ω_x is convex cone. But the inequality $l(x) - \|x\| > 0$ is equivalent to the inequality $\|x\| - l(x) < 0$, and as $l(x)$ is linear convex functional, then $\|x\| - l(x) = \varphi(x)$ is also linear convex functional. Then, according to the table (2.1) the rule (2.1), Ω_x is convex cone. By using the table (2.1) the rule (2.2), we obtain the general form of linear functional of $\Omega_x{}^*$: $\omega(x) = \alpha(l(x) - \lambda(x))$, where $\|\lambda\| \leqslant 1$.*

Example 2.5 *In the space of continuous functions $x(t)$ in the interval [0,1], we consider linear convex functional $f(x) = \max\, x(t)$. We define cone Ω_x by the inequality $f(x) < 0$. By using the rule (2.1), we can say a general form an element of the conjugate cone is, with $\omega(x) = -\alpha\mu(x)$. According to an example (1.6) $\mu(x) = \int x(t)dv$, where v is a negative measure and $\int dv = 1$. Hence, $\omega(x) = -\int x(t)dv_1$, where v_1 is a non-negative measures the cone Ω_x can be described as, a cones of the function which is negative in interval. We denote $\Omega_{1,x}$, cone of positive functionals on an interval. obviously, $\Omega_{1,x} = -\Omega_x$. Then, the general form of a linear functional in $\Omega_{1,x}{}^*$ has the form $\omega_1(t) = \int x(t)dv_1$, where v_1 is an arbitrary non-negative measure*

Example 2.6 *Let M be a closed subset of the interval [0,1]. Let*

$$f(x) = \max_{t \in M} x(t)$$

Consider the cone Ω_x , defined by the inequality $f(x) < 0$. According to rule (2.1) and example (1.6), we can easily obtain

$$\omega(x) = -\int_M x(t)dv_1\,,$$

where v_1 nonnegative M .

Example 2.7 *Consider the space of $x(t)$ is absolutely integrable functionals on the interval [0,1]. Let $\alpha(t)$ is an arbitrary bounded measurable functional not satisfying two* *inequalities* $0 \leqslant \alpha(t) \leqslant 1$ *at the same time on a set of positive measure. Consider cone* Ω_x *, defined by inequality*

$$\int_0^1 \alpha(t)x(t)dt - \int x^+(t)dt > 0$$

In order to the cone Ω_x *not to be empty, it is necessary and sufficient that linear functional*

$$l(x) = \int_0^1 \alpha(t)x(t)dt$$

It is not supporting linear convex functional $f(x) = \int x^+(t)dt$ *. But* $l(x)$ *is a supporting to* $f(x)$ *if and only if* $0 \leqslant \alpha(t) \leqslant 1$ *for almost all* t *(see. Example 1.11). Therefore, according to our assumption to* $\alpha(t)$*,* $l(x)$ *is not a supporting and thus* Ω_x *is not empty. Let*

$$\varphi(x) = \int_0^1 x^+(t)dt - \int_0^1 \alpha(t)x(t)dt$$

Obviously, $\varphi(x)$ *is linear convex functional and* $x \in \Omega_x$ *and then only when* $\varphi(x) < 0$*. According to the rule (2.2), the general form of a linear functional from there to* $\Omega_x{}^*$ *there is* $\omega(x) = \alpha[\int(\alpha(t) - \theta(t)x(t))dt]$ *, where* $0 \leqslant \theta(t) \leqslant 1$ *.*

Example 2.8 *Let* M_1 *and* M_2 *are two measurable subsets of positive measures and* $M_1 \nsubseteq M_2$*. Consider the cone* Ω_x *, which will define using inequalities*

$$x \in \Omega_x \; if \int_{M_1} x^+(t)dt - \int_{M_2} x^+(t)dt > 0$$

Let $M' = M_1 - (M_1 \cap M_2)$*. By hypothesis,* M' *is a set of positive measures. Let*

$$f_1(x) = \int_{M_1} x^+(t)\, dt, \qquad f_2(x) = \int_{M_2} x^+(t)\, dt$$

Then $f_1(x)$ *and* $f_2(x)$ *are linear convex functionals (see. Example 1.11) and* Ω_x *the cone defined by the inequality* $f_1(x) - f_2(x) > 0$*. First of all, it is clear that the cone* Ω_x *is not empty. In fact, consider the functional* $x_0(t) = X_{M'}(t)$ *, where* $X_{M'}(t)$ *is the characteristic functional of* M' *. Obviously,* $f_1(x_0) > 0$*,* $f_2(x_0) = 0$ *, and consequently,* $x_0(t) \in \Omega_x$ *. Notice, that*

Ω_x is not a convex cone. To show this, consider the case where $M_1 \supseteq M_2$, then

$$f_1(x) - f_2(x) = \varphi(x) = \int_{M'} x^+(t)dt$$

is linear convex functional. Let $x_0(t)$ be an arbitrary functional, change sign on M', when $\varphi(-x_0) > 0$, $\varphi(-x_0) > 0$ and $\varphi(x_0 - x_0) = \varphi(0) = 0$. Thus, $x_0 \in \Omega_x$,$-x_0 \in \Omega_x$ and $x_0 + (-x_0) \in \Omega_x$ We separate the cone into convex cones Ω_x , corresponding to rule (2.2). We obtain cone $\Omega_x = \bigcup \Omega_{\mu_1,x}$ where $\mu_1(x)$ is a linear supporting functional to the $f_1(x)$ and supporting to $f_2(x)$ and $x \in \Omega_{\mu_1,x,}$ if $\mu_1(x) > f_2(x)$. According to an example (1.11), general form

$$\mu_1(x) = \int_0^1 \theta_1(t)x(t)dt,$$

where $0 \leqslant \theta_1(t) \leqslant 1$ and $\theta_1(t) = 0$ if $\overline{t \in} M_1$.

A general form of a linear functional, a supporting to the $f_2(x)$ is,

$$\mu_2(x) = \int_0^1 \theta_2(t)x(t)dt,$$

where $0 \leqslant \theta_2(t) \leqslant 1$ and $\theta_2(t) = 0$ if $\overline{t \in} M_2$.

In order to $\mu_1(x)$ not to be a supporting to $f_2(x)$, it is necessary and sufficient condition that $\theta_1(t) \not\equiv 0$ on M'. We denote $\Omega_{\theta_1,x}$ with the cone which satisfies

$$\int_0^1 \theta_1(t)x(t)dt - \int_{M_2} x^+(t)dt > 0$$

where $\theta_1(t) \not\equiv 0$ on M' .

So cone Ω_x is sum of open convex cones $\Omega_{\theta_1,x}$. A general form of a linear functional $\Omega_{\theta_1,x}{}^*$:

$$\omega_{\theta_1}(x) = \alpha \int [\theta_1(t) - \theta_2(t)]x(t)dt\ ,$$

where $\theta_2(t)$ is arbitrary functional satisfying the following two conditions: where $0 \leqslant \theta_2(t) \leqslant 1$ and $\theta_2(t) = 0$ if $\bar{t} \in M_2$.

2.2. Some Type Of Functional About Variation

In this section we will consider some type of functional and some examples which help us variation. at first, we will give some definitions and theorems which has a common form.

Suppose, X be a normed space $x_0 \in X$, $\bar{x} \in X$; $f(x)$ be functional. Let

$$f'(x_0, \bar{x}) = \frac{\partial}{\partial \epsilon} f(x_0 + \epsilon \bar{x})|_{\epsilon=+0}$$

We say that the functional $f(x)$ is uniformly differentiable to the direction of the $\bar{x}_0$ if for any $\eta > 0$ there is a neighborhood $U_{\bar{x}}$ piont $\bar{x}_0$ and $\varepsilon_0 > 0$ such that $|f(x_0 + \varepsilon\bar{x}) - f(x_0) - \varepsilon f'(x_0, \bar{x})| < \eta\varepsilon$, such that for all $\bar{x} \in U_{\bar{x}}$, $\varepsilon < \varepsilon_0$. We say that the functional $f(x)$ is convex, if the following requirements holds:

(1) $|f(x_1) - f(x_2)| \leqslant c(N)\|x_1 - x_2\|$, if $\|x_1\|, \|x_2\| \leqslant N$

(2) $f(\alpha_1 x_1 + \alpha_2 x_1) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2)$, $\alpha_1, \alpha_2 > 0$; $\alpha_1 + \alpha_2 = 1$

Note that every linear convex functional is convex.We have the following theorem.

Theorem 2.3 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1981)) Every convex functional $f(x)$ is uniformly differentiable to the respect any direction.*

Theorem 2.4 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1981)) The functional $f'(x, \bar{x})$ for fixed x is linear convex respect to $\bar{x}$.*

Theorem 2.5 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1981)) Let $f(x)$ be linear convex functional $x_0 \in X$. The set of linear functionals supporting to the functional $f'(x, \bar{x})$, consists of a linear functional $\mu(x)$, the supporting to the functional $f(x)$ and satisfies $\mu(x_0) = f(x_0)$.*

Theorem 2.6 *(Dubovitskii, A.Ya. and Milyutin, A.A. (1981)) Let $f(x)$ be linear convex functional. If the set of x such that $f(x) < 0$, is not empty, then each point of the set those x for which $f(x) = 0$ is a limit for points the set $f(x) < 0$ and for the points of $f(x) > 0$.*

We now turn to examples.

Example 2.9 *Let an n-dimensional space $(\xi_1, \ldots, \xi_n)$ is given continuously differentiable functional $F(\xi_1, \ldots, \xi_n)$ and ξ^0 is fixed point. Suppose that we want to solve the problem with restrictions and find the set of prohibited variations. Obviously, $F(\xi^0 + \varepsilon\bar{\xi}) - F(\xi^0) = \varepsilon F'_{\xi}\bar{\xi} + o(\varepsilon, \bar{\xi})$ where $o(\varepsilon, \bar{\xi})$ are uniformly small for sufficiently small ε and the $\bar{\xi}$. Thus, up to small higher order increment equal to the functional value linear convex functional $\varepsilon F_{\xi'}\bar{\xi}$ (in this case, the functional linear). Thus, if $F_{\xi'}\bar{\xi} < 0$, then $\bar{\xi}$ is a prohibited variation. Assuming that $F_{\xi'} \neq 0$ at the point ξ_0 , we find that set of prohibited variations is not empty, and by Theorem (2.6) each variation $\bar{\xi}$, for which*

$F_{\xi'}\bar{\xi} = 0$, is a limit point of the variations, for which $F_{\xi'}\bar{\xi} > 0$. Hence the set of prohibited variations is completely determined by the inequality $F_{\xi'}\bar{\xi} < 0$. We see that a set of prohibited variation is corresponding with the set of type $l(\bar{\xi}) < 0$, where l is linear functional. If we looked at the problem in the maximum, then the set of prohibited variations would be determined by the inequality $F_{\xi'}\bar{\xi} > 0$

If the some problem we would restriction $F(\xi) \leqslant c$ and this would have the equality $F(\xi^0) = c$, then a lot of variations ξ , admissible this restriction would be determined entirely by the inequality $F_{\xi'}\xi < 0$.

Example 2.10 *On space of functionals $(x(t), u(t))$, defined on [0,1], where $x(t)$ is continuous functional, $u(t)$ is bounded*

measurable functional, let us define functional

$$I(x,u) = \int_0^1 f[x(t), u(t), t]dt,$$

where f is continuously differentiable functional of its variables. Let $(x_0(t), u_0(t))$ the fixed point. We find a set of prohibited variation $\bar{x(t)}, \bar{u(t)}$ on the assumption that the problem is solved to a minimum. We have

$$I(x_0 + \varepsilon\bar{x}, u_0 + \varepsilon\bar{u}) - I(x_0, u_0) = \varepsilon \int_0^1 (\frac{\partial f}{\partial x}\bar{x} + \frac{\partial f}{\partial u}\bar{u})dt + o(\varepsilon, \bar{x}, \bar{u}).$$

Here, as above, $o(\varepsilon, \bar{x}, \bar{u})/\varepsilon$ is sufficiently small and restriction ε and $\bar{x}, \bar{u}$. The derivatives $\partial f/\partial x, \partial f/\partial u$ are taken at the point $(x_o(t), u(t), t)$. As in the previous example, we will see that the increment of the functional which is determined by the linear convex functional (in this case, linear). Let's pretend that vector functional $(\partial f/\partial x, \partial f/\partial u) \not\equiv 0$. In this case the set of prohibited variation is completely determined by the inequality $\int (f_x{}'\bar{x} + f_u{}'\bar{u})dt < 0$. We see that the set of prohibited variations is cross bounded with the type $l(\bar{x}, \bar{u}) < 0$, where l is a linear functional.

For the problem to the maximum set of prohibited variations are determined by the inequality $\int (f_x{}'\bar{x} + f_u{}'\bar{u})dt > 0$.

Example 2.11 *Let the functional $f(x) = \max_t x(t)$ is given of a set of functions $x(t)$ which is continous on [0,1]. Let the functional $x_0(t)$ is fixed functional. We find a lot of prohibited variations for the minimization problem. The functional $f(x)$ is a linear convex functional and, therefore, convex functional. According to Theorem (2.3), the functional*

$f(x)$ is uniformly differentiable to the any direction. Thus, variations of $\bar{x}$ satisfying the inequality $f'(x_0,\bar{x}) < 0$, are prohibited. Lets find $f'(x_0,\bar{x}) < 0$. According to Theorem (2.5) the set of functionals subordinated $f'(x_0,\bar{x})$ coincides with the set linear functional depen to the functional $f(x)$ and there are equal $f(x)$ at $x = x_0$. According to Theorem (1.7),

$$f'(x_0,\bar{x}) = \max_{\mu} \mu(\bar{x})$$

where $\mu(x)$ is a supporting functional to the $f(x)$, and $\mu(x_0) = f(x_o)$. Let M be set of those values of t , for which $x_0(t) = f(x_0)$. Then, according to above, mentioned opinion we have

$$f'(x_0,\bar{x}) = \max \int_M \bar{x}(t)dv = \max \bar{x}(t),$$

where $dv \geqslant 0$, $\int_M dv = 1$. If for the set of $\bar{x}(t)$ for which

$$f'(x_0,\bar{x}) = \max_{t\in M} \bar{x}(t) < 0,$$

is not empty, then (see. Theorem (2.6)), the set of prohibited variations entirely determined by the inequality $f(x_0,\bar{x}) < 0$. This set is a type of set such that $\varphi(\bar{x}) < 0$, where φ is linear convex functional.

For the maximum problem for a lot of variations of $\bar{x}$ prohibited entirely defined by inequality

$$f'(x_0,\bar{x}) = \max_{t\in M} \bar{x}(t) > 0$$

The set is a set of variations of prohibited types of $\varphi(\bar{x}) > 0$, where φ is linear convex functional. This set is not convex.

Example 2.12 *In the space of continuous functionals $x(t)$, which is defined on the interval [0,1], let us define functional*

$$F(x) = \max_{t} g(\,x(t), t\,),$$

where g is continuously differentiable functional of its arguments. Let $x_0(t)$ is a fixed point. We will try to find prohibited variation for the given problem in above by using increment formula we can write

$$F(x_0 + \varepsilon\bar{x}) = \max_{t}[g(x_0(t), t) + \varepsilon g_x{}'\bar{x} + o(\varepsilon,\bar{x}) = \max[g(x_0(t), t) + \varepsilon g_x{}'\bar{x}] + o_1(\varepsilon,\bar{x}).$$

here $o(\varepsilon, \bar{x})/\varepsilon$ $\big($hence $o_1(\varepsilon, \bar{x})/\varepsilon\big)$ is small for respect sufficiently small ε and $\bar{x}$ restricted. Let us denote $g(x_0(t), t)$ by $y_0(t)$and$g_x{}'\bar{x}$ by $\bar{y}(t)$. We have thus obtained the value of the functional F a same as the linear convex functional, discussed in the preceding example, therefore, $F(x_0+\bar{x}) = f'(y, y_0)$, where f is functional, considered in the previous example. From the uniform differentiability of f it takes uniform differentiability of the functional F. Consequently, $\bar{x}$ is a prohibited variation, if $F'(x_0+\bar{x}) < 0$. Then we have,

$$f'(y_0, \bar{y}) = \max_{t \in M} \bar{y}(t)$$

(see. the previous example), then

$$F'(x_0, \bar{x}) = \max_{t \in M} g_x{}'\bar{x}(t)\,,$$

where M is the set of all values of t, for which $y_0(t) = f(y_0)$ or, equivalently, $g(x_0(t), t) = F(x_0)$. Assume that $g_x{}'$ the points of M never vanishes. It is easy to see that in this case the set prohibited variation is not empty, hence (see. Theorem 2.6) the set prohibited variations completely determined by the inequality

$$F'(x_0, \bar{x}) = \max_{t \in M} g_x{}'\bar{x}(t) < 0$$

The set of prohibited variations is the set of type of $\varphi(\bar{x}) < 0$, where φ is a linear convex functional. For the maximum problem prohibited variations is determined by the inequality

$$F'(x_0, \bar{x}) = \max_{t \in M} g_x{}'\bar{x}(t) > 0$$

This set is a set of the form $\varphi(\bar{x}) > 0$, where φ is a linear convex functional, and it is not convex.

Example 2.13 *In the space of bounded measurable functionals $u(t)$ which is defined on the interval [0,1], we consider the linear convex functional $f(u) = \operatorname{vrai}\max_t u(t)$. Let $u_0(t)$ be a fixed point and for the minimum problem let us find set of prohibited variations $\bar{u}(t)$. As it is known (see. Theorem (2.3)), the functional $f(u)$ is uniformly differentiable for each direction and $f'(u_0, \bar{u})$ is a linear convex functional for $\bar{u}(t)$; for this*

$$f'(u_0, \bar{u}) = \max_{\mu} \mu(\bar{u})$$

where the maximum is taken over all linear functionals μ supporting to $f(u)$ and coincides with $f(u)$ to $u_0(t)$ (see Theorem (1.7) and (2.5)). It is obvious that if $\bar{u}(t)$ is such that

$f'(u_0,\bar{u})<0$, then $\bar{u}(t)$ is a prohibited variation. In order to clarify the issue of the final set of prohibited variations. Let us find $f'(u_0,\bar{u})$ As we know (see. Example 2.14), in order to supporting functional $\mu(u)$ coincides with $f(u)$ at $u_0(t)$, it is necessary and sufficient to $\mu(u)=0$ if $u(t)=0$ on some $M_\delta(\delta>0)$, where the M_δ is a set consisting of all those values of t, for which the inequality $u_0(t)>f(u_0)-\delta$. Let $\bar{u}_0(t)$ is some variation. Let us find $\max\limits_{\mu}(\bar{u}_0)$ on the all supporting functional μ . We denote by $\chi_\delta(t)$ characteristic functional for sets M_δ . Obviously, $\chi_{\delta_1}(t)\geqslant\chi_{\delta_2}(t)$ if $\delta_1\geqslant\delta_2$. Let c be a constant, such that $\bar{u}_0>c$ with for all t. Then $f[\chi_\delta(t)\bar{u}_0+c(1-\chi_\delta(t))]$ is a decreasing functional δ. It is easy to see that

$$f[\chi_\delta(t)\bar{u}_0+c(1-\chi_\delta(t))]=\operatorname{vrai}\max_{t\in M_\delta}\bar{u}_0(t)$$

Let

$$m(\bar{u}_0)=\lim_{\delta\to 0}f[\chi_\delta(t)\bar{u}_0+c(1-\chi_\delta(t))]$$

then we have

$$\mu(\bar{u}_0)=\mu(\chi_\delta(t)\bar{u}_0+c(1-\chi_\delta(t)))\leqslant f(\chi_\delta(t)\bar{u}_0+c(1-\chi_\delta(t)))$$

Letting $\delta\to 0$, we obtain $\mu(\bar{u}_0)\leqslant m(\bar{u}_0)$; hence, $f'(u_0,\bar{u}_0)\leqslant m(\bar{u}_0)$. Let us now consider the functional

$$m(\bar{u})=\lim_{\delta\to 0}\operatorname{vrai}\max_{t\in M_\delta}\bar{u}(t)$$

It is easy to see that the $m(\bar{u})$ is linear convex functional. Besides, the inequality $f(\bar{u})\geqslant m(\bar{u})$ satisfy for any $\bar{u}(t)$, therefore, any linear functional $\mu(\bar{u})$, the supporting to the functional $m(\bar{u})$ is also a supporting to the functional $f(\bar{u})$. Let $\mu_0(\bar{u})$ is a linear functional, supporting to the $m(\bar{u})$ such that $\mu_0(\bar{u}_0)=m(\bar{u}_0)$. We show that $\mu_0(\bar{u})=0$ if $\bar{u}(M_\delta)\equiv 0$. In fact, $m(\bar{u})=0$ if $\bar{u}(t)$ is drawn zero for any M_δ , therefore, $\mu_0(\bar{u})\leqslant 0$ on such $\bar{u}(t)$, but the set $\bar{u}(t)$, endangered at any M_δ , is a linear manifold, the inequality $\mu_0(\bar{u})\leqslant 0$ implies that and $\mu_0(\bar{u})=0$, as required prove.But then the $\mu_0(u_0)=f(u_0)$ (see example (1.7)), therefore, $f'(u_0,\bar{u}_0)\geqslant\mu_0(\bar{u}_0)=m(\bar{u}_0)$. Finally, we obtain $f'(u_0,\bar{u}_0)=m(\bar{u}_0)$. It is easy to see that the first set of functionals $\bar{u}(t)$, satisfying the inequality $f'(u_0,\bar{u})<0$ is not empty, hence the set of prohibited variation is completely determined by the inequality $f'(u_0,\bar{u})<0$.

For the problem the maximum set of prohibited variations determined by the inequality $f'(u_0,\bar{u})>0$.

Example 2.14 *In the space of functionals $(x(t), u(t))$, where $x(t)$ is a continuous functional, $u(t)$ is bounded and measurable functional defined on the interval [0,1], we consider the functional*

$$F(x, u) = \operatorname{vrai}\max_t g(x(t), u(t), t)\,,$$

where g is a continuously differentiable functional of its arguments. Let $(x_0(t), u_0(t))$ is some point of the space. We find a set of prohibited variations for the minimum problem. We have

$$F(x_0 + \varepsilon\bar{x}, u_0 + \varepsilon\bar{u}) = \operatorname{vrai}\max_t\{g(x_0(t), u_0(t), t) + \varepsilon[g_x{}'\bar{x}(t) + g_u{}'\bar{u}(t)] + 0(\varepsilon, \bar{x}, \bar{u})\}$$

$$= \operatorname{vrai}\max_t\{g(x_0(t), u(t), t) + \varepsilon[g_x{}'\bar{x}(t) + g_u{}'\bar{u}(t)]\} + 0_1(\varepsilon, \bar{x}, \bar{u})\,,$$

where the derivatives are taken at the point $(x_0(t), u_0(t))$. Because of the continuous differentiation differentiability of g implies that there exists a functional $\eta(\varepsilon) > 0$, $\eta(\varepsilon) \to 0$ for $\varepsilon \to 0$ that $(\operatorname{vrai}\max|o(\varepsilon, \bar{x}, \bar{u})|)/\varepsilon < \eta(\varepsilon)$ for all $\bar{x}, \bar{u}$, limited by a constant ε . From $|o_1(\varepsilon, \bar{x}, \bar{u})|)/\varepsilon < \eta(\varepsilon)$ for all $\bar{x}, \bar{u}$, limited by a constant ε Let $g(x_0(t), u_0(t), t) = y_0(t)$, $g_x{}'\bar{x} + g_u{}'\bar{u} = \bar{y}(t)$. In this way $F(x_0 + \varepsilon\bar{x}, u_0 + \varepsilon\bar{u}) = f[y_0(t) + \varepsilon\bar{y}(t)] + o_1(\varepsilon, \bar{x}, \bar{u})$, where f functional considered in the example (2.13). We see that the value of the functional F is different order than ε from the value of the functional f on infinitesimal higher order. From the uniform differentiable of the functional f for each direction of the uniform differentiability of the functional F for each direction and

$$F'(x_0, u_0, \bar{x}, \bar{u}) = f'(y_0, \bar{y}) = m(\bar{y}) = m(g_x{}'\bar{x} + g_u{}'\bar{u}) = \lim_{\delta\to\,0} \operatorname{vrai}\max_{t\in M}(g_x{}'\bar{x} + g_u{}'\bar{u})$$

where M_δ is a set consisting of all those t, for which $g(x_0, u_0, t) > F(x_0, u_0) - \delta$. Suppose that $|g_{u'}| > c > 0$ for almost all t. In that event set of $\bar{x}, \bar{u}$ that satisfy the inequality $F'(x_0, u_0, \bar{x}, \bar{u}) < 0$, certainly not empty, so in this case the set of prohibited variation is completely determined by the inequality

$$F'(x_0, u_0, \bar{x}, \bar{u}) = \max(g_x{}'\bar{x} + g_u{}'\bar{u}) < 0$$

It is clear that this set is a set of type $\varphi(\bar{x}, \bar{u}) < 0$, where φ is a linear convex functional. For the maximum problem for a set of prohibited variations completely determined by the inequality $m(g_x{}'\bar{x} + g_u{}'\bar{u}) > 0$. This set is not convex.

CHAPTER 3

EXTREMUM PROBLEM WITH THE CONSTRAINTS

In this chapter by using Dubovitskiy Milyutin theorem we will try to solve such problem

Problem 3.1 *Let us extremize the functional*

$$I(x,u) = \int_{t_0}^{t_1} F(x,u,t)dt,$$

where $x(t)$ is a continuous functional defined on the interval $[t_0, t_1]$ with values from E^n; $u(t)$ is a bounded measurable functional defined on the interval $[t_0, t_1]$ with values of E^r; $F(x,u,t)$ is bounded on the any bounded set of (x,u,t), its has partial derivatives F_x and $F_{u'}$ are bounded and equicontinuous *for each fixed t by each bounded set of values (x,u,t). It is required to find $x^0(t), u^0(t)$, giving a minimum of functional I with the following restrictions:*

(1) $g(x) \leqslant 0$;

(2) $\varphi(x) \leqslant 0$;

(3) $dx/dt = f(x,u,t),\ x(t_0) = x_0$; (3.1)

(4) $x(t_1) = x_1$.

There $g(x), \varphi(x)$ are continuously differentiable functionals of their arguments; $g_x{}' \neq 0$, if $g(x) = 0$; $\varphi_{u'} \neq 0$; if $\varphi(u) = 0$. The functional $f(x,u,t)$ is bounded on each bounded set (x,u,t), has partial $f_x{}'$ and $f_u{}'$, which is bounded and equicontinuous for each fixed on each bounded t the set of values (x,u,t), and takes the value of E^n. We assume and that $g(x_0) \leqslant 0$ and $g(x_1) \leqslant 0$. In chapter one it is considered some difficult problem. Problem rather than limiting $\varphi(u) \leqslant 0$ required to value u and belonged to a set of E^r, and time t_1 is fixed. Here we consider the problem in this formulation. However, it seems *more helpful to consider first simple task, in which the lower auxiliary constructions and all the essential features of the method are the clear.*

Let us analyze the problem. In a space of variation we introduce the space $\overline{W}$ pairs of functionals $\bar{x}(t), \bar{u}(t)$, where $\bar{x}(t)$ is continuous on the interval $[t_0, t_1]$ and takes values from E^n,$\bar{u}(t)$ is bounded and measurable on the interval $[t_0, t_1]$ and takes the value of E^r.

1. Let us investigate the structure of the set of prohibited variations. Such type of functionals we have seen in example (2.10) we considered the functional with the values of E^1, instead of E^n and E^r, which these opinion does not change the substance of the matter. Thus, the set of prohibited variations is not empty and it coincides with the set of variations of $\bar{x}(t)$ and $\bar{u}(t)$, that satisfy inequality

$$\int_{t_0}^{t_1} [F_x{}'(x^0, u^0, t)\bar{x} + F_u{}'(x^0, u^0, t)\bar{u}]\, dt < 0$$

Assume first that the $F_x{}'(x^0, u^0, t)$ or $F_u{}'(x^0, u^0, t)$ is not identically zero. In this case the set of prohibited variations not empty and a convex open cone in $\overline{W}$, given one linear inequality. We denote it by Ω_0

1^0. We now investigate to the restriction $g(x) \leqslant 0$. If $g[x^0(t)] < 0$ for every t in the interval $[t_0, t_1]$, the set of variations which is prohibited for this restriction that coincides with the whole space. Suppose that $g[x^0(l)] = 0$ on a set of M values of t. In this case, the set of prohibited variations coincides with the set of prohibited variations in the problem at the minimum of the functional

$$\varphi(x, u) = \max_t g(x).$$

This functional has been considered in example 2.12. In the case of $g_x{}'[x^0(t)] \neq 0$ for $t \in M$, the set of prohibited variations in the problem on minimum coincides with many variations in $\bar{x}(t), \bar{u}(t)$, satisfying inequality $g_x{}'[x^0(t)]\bar{x}(t) < 0$ for $t \in M$, consequently, the set of admissible variations to limit $g(x) \leqslant 0$ defined by the inequality

$$\max_{t \in m} g_x{}'[x^0(t)]\bar{x}(t) < 0$$

Thus the set of admissible variations is not empty open convex cone in the space $\overline{W}$, given the inequality $r(\bar{x}, \bar{u}) < 0$, where $r(\bar{x}, \bar{u})$ linear convex functional. Let this cone through Ω_1 .

2^0. If

$$\text{vrai}\max_t \varphi[u^0(t)] < 0$$

the set of variations of $\bar{x}(t)$, $\bar{u}(t)$, which is admissible to restrict the $\varphi(u) \leqslant 0$ is the whole space. If $\text{vrai}\max_t \varphi[u^0(t)] = 0$, then set of allowable variation coincides with the set of prohibited variation for the following minimum problem

$$\sigma(x,u) = \text{vrai}\max_t \varphi(u).$$

Functional of this type has been considered in example (2.14). Let $\delta > 0$. We define the set of M_δ values of t with following way: $t \in M_\delta$ if and only if $\varphi[u^0(t)] > -\delta$. In the case where $|\varphi_u{}'| > \beta > 0$ on the set M_δ , the set of prohibited variations as we have seen, is determined by the inequality $m(\bar{x},\bar{u}) < 0$, where

$$m(\bar{x},\bar{u}) = \lim_{\delta\to 0} \text{vrai}\max_{t\in M_\delta} \varphi_u{}'[u^0(t)]\bar{u}(t).$$

Thus, the set of prohibited variations determined by the inequality $m(\bar{x},\bar{u}) < 0$. Since $m(\bar{x},\bar{u})$ is a linear convex functional and set of prohibited variations is not empty (by assumption $\varphi_u{}'[u^0(t)] \neq 0$ if $\varphi(u) = 0$), the inequality $m(\bar{x},\bar{u}) < 0$ defines open convex cone in the space of pairs of $\bar{x}(t), \bar{u}(t)$. Let us denote this cone by Ω_2 .

3^0 We now turn to the study of inequality constraints. We have two types of restrictions (see. (3.1), restrictions (3) and (4)). The first one is in that a pair of $x(t), u(t)$ is related by the equation $dx/dt = f(x,u,t)$ with the initial conditions $x(t_0) = x_0$. The second restrictions is that that $x(t_1) = x_1$ at $t = t_1$. It is easy to find a lot of admisible variations for each of these restrictions. Indeed, if $\bar{x}(t), \bar{u}(t)$ is riding variation of the first of these constraints (constraint 3)), then, by definition, there is a sequence $\varepsilon_n, \tilde{x}_n, \tilde{u}_n$, tends to zero as $n \to \infty$, such that

$$\frac{d}{dt}[x^0 + \varepsilon_n(\bar{x} + \tilde{x}_n)] = f[x^0 + \varepsilon_n(\bar{x} + \tilde{x}_n), u^0 + \varepsilon_n(\bar{u} + +\tilde{u}_n), t],$$

$$[x^0 + \varepsilon_n(\bar{x} + \tilde{x}_n)]|_{t=t_0} = x_0$$

Expanding the powers of, and the left and right-hand side, we obtain

$$\frac{d\bar{x}}{dt} = f_x{}'\bar{x} + f_u{}'\bar{u}, \qquad \bar{x}(t_0) = 0, \tag{3.2}$$

where $f_x{}'$ and $f_u{}'$ are taken at the points $x^0(t), u^0(t), t$. On the other hand,each variation of $\bar{x}(t), \bar{u}(t)$, satisfying the system (3.2) is admissible variation. To see this, it is enough to consider solution of (3.1) with $u(t) = u^0(t) + \varepsilon\bar{u}(t)$ and $x(t_0) = x_0$. Setting $x(t) = x^0(t) + \varepsilon(\bar{x} + \tilde{x})$, and using the theorems on differentiation solutions to the parameter that $\tilde{x} \to 0$ as $\varepsilon \to 0$. Thus,set of admissible variations of $\bar{x}(t), \bar{u}(t)$ of the first restriction is entirely determined by the requirement that $\bar{x}(t), \bar{u}(t)$ satisfies the system (3.2). Many variations satisfying the system (3.2) is a subspace in the space $\overline{W}$. We denote it by L_1. The second restriction (4) investigated quite simple. If the variation of $\bar{x}(t), \bar{u}(t)$ is valid this restriction, it is obvious that $\bar{x}(t_1) = 0$. This condition completely characterizes the variation admissible for the second restriction. Variations of $\bar{x}(t), \bar{u}(t)$, satisfying the condition $\bar{x}(t_1) = 0$, form a subspace in the space $\overline{W}$. We denote it by L_2. It's obvious that variation of $\bar{x}(t), \bar{u}(t)$, admissible respect restriction and satisfy the system

$$\frac{d\bar{x}}{dt} f_x{}'\bar{x} + f_u{}'\bar{u}, \qquad \bar{x}(t_0) = 0, \qquad \bar{x}(t_1) = 0. \tag{3.3}$$

There is, however, question whether any variation of $\bar{x}(t), \bar{u}(t)$, satisfying system (3.3) is admissible for the restriction (3) and (4). Here sufficient conditions which guarantee that opinion. Let $\psi(t)$ is a solution of the system

$$-\frac{d\psi}{dt} = (f_x{}')^{*}\psi(t), \tag{3.4}$$

determined at $[t_0, t_1]$. Under $(f_x{}')^{*}$ understand the transpose of the matrix $f_x{}'$. In order for any variation of $\bar{x}(t), \bar{u}(t)$, satisfying the system (3.3), it would be admissible respect restrictions of intersection (3) and (4) it is sufficient existence r-dimensional vector functional $(\psi, f_u{}') \not\equiv 0$ which is satisfies

$$(\psi, f_u{}') = \sum_{i=1}^{n} \psi_i(t)(f_i)u'.$$

Further we will call this condition as the condition of nonsingular system (3.1) in the neighborhood of $x^0(t), u^0(t)$. The proof of this statements can be found in [1]. Let us first assume that there is a non-degeneracy condition; then set of variations allowed for restrictions (3) and (4), fully characterized by a system (3.3). set of variations satisfying System (3.3) is a subspace, which we denote by L. Obviously, $L = (L_1 \cap L_2)$.

So, we looked at each constraint separately and found the structure of the sets of prohibited and admissible variations. In this case all these sets are convex cones. We now find each obtained from cones general form of a linear functional in the dual cone.

2. Since the cone Ω_0 is defined by a linear inequality

$$\int_{t_0}^{t_1} (F_x{}'\bar{x} + F_u{}'\bar{u})dt < 0\,,$$

then, in corresponding to the rule (2.1) and example (2.1), the general form of linear functional is $\Omega_0{}^*$

$$-\alpha \int_{t_0} (F_x{}'\bar{x} + F_u{}'\bar{u})dt.$$

1^0. Let us find the general form of linear functional $\Omega_1{}^*$. Cone Ω_1 is defined by the inequality

$$r(\bar{x}, \bar{u}) = \max_{t \in M} g_x{}'\bar{x} < 0\,,$$

where $r(\bar{x}, \bar{u})$ is a linear convex functional. In order to find a general form of linear functional, belonging to $\Omega_1{}^*$, it is necessary, corresponding to rule (2.1) (a cone Ω_1 is not empty), to find primarily general form linear functional supporting to the functional $r(\bar{x}, \bar{u})$. We use the following method. Assume $A(\bar{x}, \bar{u}) = g_x{}'\bar{x} = \bar{y}(t)$. Consider a bounded linear operator A mapping the space of pairs of $\bar{x}(t), \bar{u}(t)$ in the space of continuous functionals on the interval $[t_0, t_1]$, it is clear,

$$r(\bar{x}, \bar{u}) = \max_{t \in M} \bar{y}(t) = r_1(\bar{y}).$$

Linear convex functional $r_1(\bar{y})$ is considered by the example (1.6). General form of a linear functional, a supporting to the functional $r_1(\bar{y})$, is

$$\int_{t_0} \bar{y}(t)dv$$

where the measure v has the following properties: $dv \geqslant 0$; $\int_{t_0}^{t_1} dv = 1$; measure v and concentrated on the set M.

Hence, in according to rule (1.11), we obtain the general form of a linear functional supporting to the $r(\bar{x}, \bar{u})$, is

$$-\int_{t_0}^{t_1} g_x{}'\bar{x}dv,$$

where the measure v has the properties listed above. Using rule 5.1, we find that the general form of linear functional belonging $\Omega_1{}^*$, is

$$-\alpha \int_{t_0}^{t_1} g_x{}'\bar{x}dv, \qquad \alpha \geqslant 0\,.$$

Put $\alpha v = \mu$, then any linear functional belonging to $\Omega_1{}^*$, can be written in the form

$$-\int_{t_0}^{t_1} g_x{}'\bar{x}d\mu,$$

where the measure μ has the following two properties: $d\mu \geqslant 0$, the measure μ focus on the set M.

2^0. We now find the general form of linear functional Ω_2*. As the cone Ω_2 is not empty and is given by the inequality

$$m(\bar{x},\bar{u}) = \lim_{\delta\to 0} \text{vrai} \max_{t\in M_\delta} \varphi_u{}'\bar{u} < 0 \qquad then$$

$m(\bar{x},\bar{u})$ is a linear convex functional, then to find the general form of a linear functional in $\Omega_2{}^*$, it is necessary first to find the general form linear functional supporting to the $m(\bar{x},\bar{u})$ (see. Rule (2.1)). Let us use the following method:

Putting $A(\bar{x},\bar{u}) = \varphi_u{}'\bar{u} = \bar{y}$. Let us introduce a linear bounded operator A, a mapping space pairs $\bar{x}(t),\bar{u}(t)$ into the space bounded measurable functionals defined on $[t_0,t_1]$,

$$m_1(\bar{y}) = \lim_{\delta\to 0} \text{vrai} \max_{t\in M_\delta} \bar{y},$$

Then $m(\bar{x},\bar{u}) = m_1(\bar{y})$. Let us find the general form of linear functional supporting to the functional $m_1(\bar{y})$. For this we introduce another linear convex functional:

$$s(y) = \text{vrai} \max_{t\in[t_0,t_1]} y.$$

Let $y^0(t) = \varphi(u^0)$. According to example (1.5) $m_1(\bar{y}) = s'(y^0,\bar{y})$. The functional $s'(y^0,\bar{y})$ has been investigated in detail in the same example. A general form of a linear functional, a supporting to $s'(y^0,\bar{y})$, there is $l(\bar{y})$, where functional l has the following properties: $l(\bar{y}) \geqslant 0$ if $y \geqslant 0$. About almost everywhere; $l(\bar{y} \equiv 1) = 1$; $l(\bar{y}) = 0$ if $\bar{y} = 0$ almost everywhere on M_δ.

By using the rule (1.11), it is not difficult to find a general form of a linear functional, a supporting to the functional $m(\bar{x},\bar{u})$. We find that the general form linear functional supporting to the functionals $m(\bar{x},\bar{u})$ is a $l(\varphi_u{}'\bar{u})$. Hence (see. rule (2.1)) the general form of linear functional belonging $\Omega_2{}^*$, is $-\alpha l(\varphi_u{}'\bar{u})$ where $a \geqslant 0$. Let $\alpha l(\varphi_u{}'\bar{u}) = \lambda(\bar{u})$. It is clear

but $\lambda(\bar{u})$ is completely characterized by the following properties: $\lambda(\bar{u}) \geqslant 0$ if $\varphi_u{}'\bar{u} \geqslant 0$ almost everywhere; $\lambda(\bar{u}) = 0$ if $\varphi_u{}'\bar{u} = 0$ almost everywhere on M_δ

Thus, the general form of linear functional belonging $\Omega_2{}^*$, is $-\lambda(\bar{u})$.

3^0. Now we find the set of linear functionals L^*. Since L is subspace then linear functionals of the L^* vanish on L. We use the fact that $L = (L_1 \bigcap L_2)$. Since there L_2 there is ∞ $-n-$ dimensional subspace of pairs $\bar{x}(t), \bar{u}(t)$, then, according to Theorem (1.6), $L^* = L_1{}^* + L_2{}^*$. A general form of a linear functional of $L_1{}^*$, according to theorem (1.5), is a $l[\bar{x} - x'(\bar{u})]$ where $x'(\bar{u})$ satisfy the system (3.2). (7.2). It is not easy to see that general form of a linear functional $L_2{}^*$ is $(c\bar{x})_{t=t_l}$,

$$(c\bar{x}) = \sum_{i=1}^{n} c_i \bar{x}_i,$$

where c_i is fixed numbers, but the general form of linear functional L^* is $l[\bar{x} - x'(u)] + (c\bar{x})_{t=t_1}$. We have found, therefore, a general form of the linear functional, within the cones dual cone obtained by varying problems.

Euler equation. A necessary condition for an extremum is that there are linear functionals of each one of the conjugate cone, not all simultaneously zero but their sum is equal to zero. Euler equation, therefore, has the form

$$-\alpha \int_{t_0}^{t_1} (F_x{}'\bar{x} + F_u{}'\bar{u})dt - \int_{t_0}^{t_1} g_x{}'\bar{x}d\mu - \lambda(\bar{u}) + l[\bar{x} - \bar{x}'(\bar{u})] + (c\bar{x})_{t=t_1} = 0\,.$$

The equality holds for all $\bar{x}(t), \bar{u}(t)$, and $\alpha, \mu, \lambda \neq 0$ at the same time we investigate the Euler equation. Let take $\bar{x}(t) = x'(\bar{u})$ then we can obtain

$$-\alpha \int_{t_0}^{t_1} (F_x{}'x'(\bar{u}) + F_u{}'(\bar{u}))dt - \int_{t_0}^{t_1} g_x{}'x'(\bar{u})d\mu - \lambda(\bar{u}) + (cx'(\bar{u}))_{t=t_1} = 0\,.$$

This expression can be written as follows:

$$\int_{t_0}^{t_1} [(\psi, f_u{}') - \alpha F_u{}']\bar{u}dt - \lambda(\bar{u}) = 0\,.$$

where $\psi(t)$ satisfies

$$-\frac{d\psi}{dt} = (f_x)^*\psi - \alpha F_x{}' - g_x{}'\frac{d\mu}{dt}\,, \qquad \psi(t_1) = c. \tag{3.5}$$

Here $d\mu/dt$ is understood in the sense of generalized functionals. Finally we get

$$\lambda(\bar{u}) = \int_{t_0}^{t_1} [(\psi(t), f) - \alpha F]_{u'}\bar{u}dt\,,$$

First of all show that ψ and is not identically equal to zero at the same time

From $\psi \equiv 0$ and $\alpha \equiv 0$, it follows that the measure $\mu \equiv 0$, on the set M, which focuses measure μ, does not include the endpoints $[t_0, t_1]$, and $\lambda(\bar{u}) \equiv 0$. But then α, u, and λ vanishes simultaneously, contradicts the hypothesis. Let $H(t, u) = (\psi(t), f) - \alpha F$. The Euler equation can then be written as

$$\int_{t_0}^{t_1} \frac{\partial H}{\partial u} \bar{u} dt = \lambda(\bar{u}).$$

This equation implies that $\partial H/\partial u = 0$ almost everywhere on any a set of positive measure on which $\varphi(u^0) < \gamma < 0$. In fact, $\lambda(\bar{u}) = 0$, whatever the functional $\bar{u}(t)$, focused on such set. Hence it is easy to get that $\partial H/\partial u = 0$ almost everywhere on any set where $\varphi(u^0) < 0$, consequently, any set positive measure on which $|\partial H/\partial u| > 0$, $\varphi(u^0) = 0$ almost everywhere. Finally, if $\bar{u}(t)$ is such that $\varphi_u{}' \bar{u} < 0$ for any M_δ then $(\partial H/\partial u)\bar{u} \leqslant 0$ almost everywhere on this set. This fact implies from the fact that $-\lambda(\bar{u})$ in this case is a non-negative value. Easily see also that if $\partial H/\partial u$ possesses all the above properties, then

$$-\int_{t_0}^{t_1} \frac{\partial H}{\partial u} \bar{u} dt$$

is a functional, included in ${\Omega_2}^*$. Thus, a necessary condition extremum is that there is a measure μ concentrated on M, a functional ψ, satisfying the system (3.5), and is $\alpha \geqslant 0$, that $\partial H/\partial u$ has the above properties. These necessary conditions we obtained under the assumption that the cone Ω_0 is not empty and that System (3.1) is non-degenerate in the neighborhood of the solution $x^0(t), u^0(t)$. We show In these cases, the formulation of the necessary conditions in the term of $H(t, u)$ remains the same:

a) The cone Ω_0 is empty. This means that, no matter what variation of $\bar{x}(t), \bar{u}(t)$

$$\int_{t_0}^{t_1} (F_x{}' \bar{x} + F_u{}' \bar{u}) dt = 0\,,$$

therefore, $F_x{}' \equiv 0$, $F_u{}' \equiv 0$ Suppose in this case $\alpha = 1, \mu \equiv 0, \psi \equiv 0$. Obviously, $\psi \equiv 0$ satisfies the system (3.5). In this case, $H = -F$ and $\partial H/\partial u = 0$ almost everywhere;

b) the system (3.1) is singular the neighborhood of $x^0(t), u^0(t)$. In this case there is a nontrivial solution $\psi(t)$ of (3.4) that $(\psi f_u{}') = 0$ almost everywhere. Let $\alpha = 0$ and $\mu = 0$, then ψ is also a solution of (3.5). In this case, $H = (\psi, f)$ and is easily seen that $\partial H/\partial u = 0$ almost everywhere.

Problem 3.2 *Consider the functional*

$$I(x,u) = \max_t g(x), \qquad t_0 \leqslant t \leqslant t_1.$$

Assume that $x^0(t)$, $u^0(t)$, t_1 provide at least one functional $I(x,u)$ with the following limitations:

(1) value of $u(t)$ belong to some set D of space E^r;

(2) $dx/dt = f(x,u)$, $x(t_0) = x_0$;

(3) $x(t_1) = x_1$.

Regarding the functional $g(x)$ assume continuous differentiability with respect to x. Let $g^0 = I(x^0,u^0)$. We assume that $g_{x'} \neq 0$ if $g(x) = g^0$. Furthermore, for simplicity we require $g(x_0) < g^0$, $g(x_1) < g^0$.

Assumptions regarding the functionals $x(t), u(t)$ and $f(x,u)$ are the same (2). Reduction this problem to problem 1 by using control $v(\tau)$ is made the same way as in the problem (2). In this case prohibited variation now the Ω_1 cone problem (1). The rest of the cones remain unchanged. In the Euler equation (3) will therefore one member less than the Euler equation (2). We obtain

$$-\int_0^1 g_x{}'\bar{x}\, d\mu_\tau - \lambda(\bar{v}) + l[\bar{x} - x'(\bar{v})] + (c,\bar{x})_{\tau=1} = 0\,,$$

where μ_τ and λ have the same meaning as in task 2, and does not vanish simultaneously. Repeating the arguments that we have used to address Problem 2, we obtain the following maximum principle:

There exists a measure $\mu \geqslant 0$, concentrated on the set of values t , for which the $g[x(t)] = g^0$ then $(\psi(t), f[x^0(t), u(t)]) \leqslant 0$ and equality holds for almost all t if $u(t) = u^0(t)$. The functional $\psi(t) \not\equiv 0$ and satisfies

$$-\frac{d\psi}{dt} = (f_x{}') * \psi - g_x{}'\frac{d\mu}{dt}\,.$$

Problem 3.3 *Given functional*

$$I(x,u) = \operatorname*{vrai\,max}_{t_0\leqslant t\leqslant t_1} g[x(t),\; u(t)].$$

The functional $x(t)$ is continuous on the interval $[t_0, t_1]$ and takes values from E^n. The functional $u(t)$ is measurable, bounded, with values of E^n. Let $x^0(t)$, $u^0(t)$, t_1 give at least functional $I(x,u)$ under the following restrictions::

(1) $dx/dt = f(x,u)$, $x(t_0) = x_0$;

(2) $x(t_1) = x_1$.

Regarding the functional $g(x,u)$ and the functionals $f(x,u)$ we assumed continuously differentiable in both arguments. Required to find conditions that are satisfied by $x^0(t), u^0(t), t_1$. put $g^0 = I[x^0(t), u^0(t)]$. We assume for simplicity that each $x(t)$ the set of those values of $u(t)$, for which $g(x,u) \leqslant g^0$, restricted $g_{u'} \neq 0$ in the points $x(t), u(t)$, where $g(x,u) = g^0$. To obtain corresponding maximum principle, as well as in problems 2 and 3, we introduce a new control $v(\tau)$. However, in this problem we will be considered along with the control $v(\tau)$ and control $u(\tau)$. Here design, by which we move from task to task (4) Type (1), i.e. to the problem in which we confine ourselves to small variations or what is the same, considering the weak extremum. Let $v^0(\tau)$ be a functional that takes two values, one of which is zero, and

$$\int_0^1 v^0(\tau)d\tau = t_1 - t_0 .$$

We will also assume that a zero value is taken on the system intervals. put

$$t = t_0 + \int_0^\tau v^0(\tau)d\tau, \qquad x^0(\tau) = x^0[t(\tau)].$$

We define the functional $\tilde{u}^0(\tau)$ as follows:

$\tilde{u}^0(\tau) = u^0[t(\tau)]$, if $v^0(\tau) \neq 0$. At each interval, where $v^0(\tau) = 0$, we assume that $\tilde{u}^0(\tau)$ takes on a constant value. These values we chosen arbitrarily with only one restriction $g[x^0(\tau), \tilde{u}^0(\tau)] < g^0$ (on any interval where $v^0(\tau) = 0, x^0(\tau)$ is constant). Obviously, the triple functionals $x^0(\tau), \tilde{u}^0(\tau), v^0(\tau)$ is a point of minimum of functional

$$I'(x,u,v) = \operatorname*{vrai\,max}_{0\leqslant\tau\leqslant 1} g[x(c), u(\tau)]$$

with constraints

1′**)** $dx/d\tau = v(\tau)f[x(\tau), u(\tau)], \qquad x(0) = x_0$;

2′**)** $x(1) = x_1$;

3′**)** $v(\tau) \geqslant 0$.

We will not give here the proof of this assertion, since we conducted a proof of example which is closer to the Problem (2). In what follows we consider, as well as Problem (1), small variations of the functionals $x(\tau)u(\tau), v(\tau)$, but since $v(\tau)$ included linearly, the local conditions necessary for the minimum of the functional $I'(x, u, v)$ will give the maximum principle for the functional $I(x, u)$, similar to just as it was in problems (2) and (3). We find the cone prohibited variations. Functional of the same type, and functionality that $I(x, u)$, we examined Example (2.14). Since $g_u{}' \neq 0$, if $g(x, u) = g^0$, the set prohibited variation $\bar{x}(\tau), \bar{u}(\tau), \bar{v}(\tau)$ is not empty and is given the inequality $m(\bar{x}, \bar{u}, \bar{v}) < 0$, where m is linear convex functional that we introduced in (2.13). Recall that

$$m(\bar{x}, \bar{u}, \bar{v}) = \lim_{\delta \to 0} \text{vrai} \max_{\tau \in M_\delta} (g_x{}'\bar{x} + g_u{}'\bar{u}),$$

where M_δ be the set of all values of τ, such that $g[x^0(\tau), u^0(\tau)] > g^0 - \delta$, hence the set of prohibited variations is open convex cone, which we denote by Ω_0. Let us find the general form of a linear functional from $\Omega_0{}^*$. For this rule (2.1) to find the general form of linear functional reference to the functional m. Using arguments similar to the arguments in paragraph. (2^0) for the problem (1), we find that the linear functional $\mu(\bar{x}, \bar{u}, \bar{v})$, the supporting to the functional m is completely determined by the following properties:
$\mu(\bar{x}, \bar{u}, \bar{v}) = \mu'(g_x{}'\bar{x} + g_u{}'\bar{u})$, where μ' be linear functional in the space bounded measurable functionals defined on the interval [0,1]; $\mu'[\bar{y}(\tau)] \geqslant 0$ if $\bar{y}(\tau) \geqslant 0$; $\mu'[\bar{y}(\tau) \equiv 1] = 1$; $\mu'[\bar{y}(\tau)] = 0$, if $\bar{y}(\tau) = 0$ almost everywhere at any M_δ

According to Rule (2.1), the general form of linear functional has $\Omega_0{}^*$ if $\alpha\mu'(g_x{}'\bar{x} + g_u{}'\bar{u})$, where $\alpha \geqslant 0$. Restrictions (1′) and (2′) are investigated completely same as well as in the problem 1 is investigated restriction (3) and (4). At first we assume that the system (3.1) is non-degenerate in the neighborhood of the solution $x^0(\tau), \tilde{u}^0(\tau), v^0(\tau)$. In this case, the set of admissible variations of $\bar{x}(\tau), \bar{u}(\tau), \bar{v}(\tau)$ is defined by the following equations:

$$\frac{d\bar{x}}{d\tau} = v^0(\tau)(f_x{}'\bar{x} + f_x{}'\bar{u}) + \bar{v}(\tau) f[x^0(\tau), \tilde{u}^0(\tau)], \tag{3.6}$$

$$\bar{x}(0) = \bar{x}(1) = 0.$$

As can be seen from (3.6), admissible variations restrictions on (1′) and (2′), form a subspace of $\bar{x}(\tau), \bar{u}(\tau), \bar{v}(\tau)$. Let us denoted it by L. A general form of a linear functional of L^* is $l[\bar{x} - \acute{x}(\bar{u}, \bar{v})] + (c\bar{x})_{\tau=1}$ (see problem 1), where $x'(\bar{u}, \bar{v})$

satisfies

$$\frac{dx'}{d\tau} = v^0(\tau)(f_x{}'\bar{x} + f_u{}'\bar{u}) + \bar{v}(\tau)f[x^0(\tau), \tilde{u}^0(\tau)], \qquad x'(\tau = 0) = 0.$$

Restriction (3′) varies in the same manner as in task (2′). A linear functional from the corresponding conjugate cone is denoted by $\lambda(\bar{v})$. Recall that the functional λ properties characterized by $\lambda(\bar{v}) \geqslant 0$ if $\bar{v} \geqslant 0$; $\lambda(\bar{v}) = 0$ if $\bar{v} = 0$ everywhere where $v^0(\tau) = 0$.

Euler's equation:

There exist $\alpha \geqslant 0 \quad \mu', l, c, \lambda$ that

$$-\alpha\mu'[g_x{}'\bar{x} + g_u{}'\bar{u}] + l[\bar{x} - x'(\bar{u}, \bar{v})] + (c\bar{x})_{\tau=1} + \lambda(\bar{v}) = 0, \tag{3.7}$$

wherein α and λ are not zero simultaneously. Equation 3.7 holds for all $\bar{x}, \bar{u}, \bar{v}$. Let $\bar{x} = x'(\bar{u}, \bar{v})$ and write separately equality with the first $\bar{u}$ and $\bar{v}$. We get

$$\begin{aligned} -\alpha\mu'[g_x{}'x'(\bar{u}) + g_u{}'\bar{u}] + (cx'(\bar{u})\,)_{\tau=1} = 0 \\ -\alpha\mu'[g_x{}'x'(\bar{v})] + (cx'(\bar{v}))_{\tau=1} + \lambda(\bar{v}) = 0 \end{aligned} \tag{3.8}$$

Where $x'(\bar{u})$ (respectively, $x'(\bar{v})$) is a solution of 3.8 with $\bar{v}$ (respectively, $\bar{u}$) equal to zero. Now let $R(\tau)$ be an arbitrary functional with values in E^n, and $x'(R)$ is a solution of system of equations

$$\frac{dx'}{d\tau} = v^0(\tau)f_x{}'x' + R, \qquad x'(\tau = 0) = 0\,.$$

We have

$$\lambda[g_x{}'x'(R)] = \int_0^1 (\psi, R)d\tau,$$

where $\psi(\tau)$ be a bounded functional with values in E^n, clearly defined According to the equation. In fact, it is easy to verify that the expression $-\alpha\mu'[g_x{}'x'(R)]$ is a continuous normal L_1 linear functional by R.

Considering the terms $v(\tau)$, we obtain

$$\int_0^1 (\psi, f)\bar{v}d\tau + (cx'(\bar{v}))_{\tau=1} + \lambda(\bar{v}) = 0,$$

or

$$\int_0^1 (\psi + \psi_1, f)\bar{v}d\tau + \lambda(\bar{v}) = 0, \tag{3.9}$$

where ψ_1 satisfies the system of equations

$$-\frac{d\psi_1}{d\tau} = v^0(\tau)(f_x{}')^*\psi_1, \qquad \psi_1(\tau = 1) = c.$$

Considering the terms with $\bar{u}(\tau)$, we obtain

$$\int_0^1 v^0(\tau)(\psi + \psi_1, f_u{}')\bar{u}d\tau = \alpha\mu'(g_u{}'\bar{u}), \tag{3.10}$$

$$\bar{u} = \frac{g_u{}'}{|g_u{}'|^2} g_x{}'x\prime(R)$$

This equality obviously makes matter on some M_δ, namely, M_δ is where $|g_u{}'| > \beta > 0$, where β be a constant. Other form functional for $\bar{u}$ of τ tons due to the functional properties of μ' does not matter. In this way,

$$\alpha\mu'[g_x{}'x'(R)] = \int_0^1 v^0(\tau)(\psi + \psi_1, f_u{}')\frac{g_u{}'}{|g_u{}'|^2} g_x{}'x'(R)d\tau \ .$$

But

$$\alpha\mu'[g_x{}'x'(R)] = -\int_0^1 (\psi, R)d\tau \ ,$$

therefore, for any absolutely integrable on [0,1] functional $R(\tau)$ we have the equality

$$\int_0^1 (\psi, R)d\tau = -\int_0^1 v^0(\tau)(\psi + \psi_1, f_u{}')\frac{g_u{}'}{|g_u{}'|^2} g_x{}'x'(R)d\tau \ .$$

But then the ψ is a solution of

$$-\frac{d\psi}{d\tau} = v^0(\tau)(f_x{}')^*\psi - v^0(\tau)[(\psi + \psi_1, f_u{}')\frac{g_u{}'}{|g_u{}'|^2} g_x{}'], \qquad \psi(1) = 0$$

Now we put $\psi + \psi_1 = \psi_0$, we obtain

$$-\frac{d\psi_0}{d\tau} = v^0(\tau)(f_x{}')^*\psi_0 - v^0(\tau)[(\psi_0 + f_u{}')\frac{g_u{}'}{|g_u{}'|^2} g_x{}'], \qquad \psi_0(1) = c$$

In addition, we have (see. (3.9))

$$\int_0^1 (\psi_0, f)\bar{v}d\tau + \lambda(\bar{v}) = 0.$$

Let $H(\tau) = (\psi_0(\tau), f(x^0(\tau), \tilde{u}^o(\tau)))$ As

$$\int_0^1 H(\tau)\bar{v}d\tau + \lambda(\bar{v}) = 0.$$

then $H(\tau) = 0$ almost everywhere on the set of τ for which $v^0(\tau) \neq 0$, and $H(\tau) \leqslant 0$ for those t for which $v^0(\tau) = 0$. Let $\psi(t) = \psi_0[\tau(t)]$. Then $\psi_0(\tau)$ be a continuous functional of τ and does not change at any interval, where $v_0(\tau) = 0$, the $\psi(t)$ is determined by this equation is uniquely and is a continuous functional of t. Obviously, $\psi(t)$ satisfies system

$$-\frac{d\psi}{dt} = (f_x{}')^*\psi - (\psi, f_u{}')\frac{g_u{}'}{|g_u{}'|^2}g_x{}' \tag{3.11}$$

Let $\pi(t,u) = (\psi(t), f[x^0(t), u])$ then $\pi[t, u^0(t)]$ almost everywhere equal zero, and $\pi(t_k, u_k) \leqslant 0$. Recall that t_k is the image of the interval, at which $v^0(\tau) = 0$;

$u_k = \tilde{u}^0(\tau)$ in this interval. Now choose the point t_k, u_k, so that $g[x^0(t_k), u_k] < g_0$, and any point t, u and such that $g[x^0(t), u] < g^0$, was to limit to t_k, u_k.

We obtain the following maximum principle:

There exists a functional $\psi(t)$ satisfies the system 3.11 that

$$\pi(t,u) = (\psi(t), f[x^0(t), u]) = 0 \tag{3.12}$$

almost everywhere when $u = u^0(t)$ and $\pi(t,u) \leqslant 0$ for the t, u, for which $g[x^0(t), u] < g^0$.

CONCLUSION

In this thesis we solved optimal control problem

$$I(x,u) = \int_{t_0}^{t_1} F(x,u,t)dt$$

with constraints. We obtained necessary optimality condition for this problem.

In future this problem can be extended to the problem which has minimization functional

$$I(x,u) = s(x(t_1), u(t_1)) + \int_{t_0}^{t_1} F(x,u,t)dt,$$

and some extra constraints $g(x,u) \leqslant 0$. For this problem it can be consider necessary optimality condition.

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