International Journal of Control

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

Necessary conditions for partially observed optimal control of general McKean–Vlasov stochastic differential equations with jumps

Hakima Miloudi, Shahlar Meherrem, Imad Eddine Lakhdari & Mokhtar Hafayed

To cite this article: Hakima Miloudi, Shahlar Meherrem, Imad Eddine Lakhdari & Mokhtar Hafayed (2021): Necessary conditions for partially observed optimal control of general McKean–Vlasov stochastic differential equations with jumps, International Journal of Control, DOI: 10.1080/00207179.2021.1961020

To link to this article: https://doi.org/10.1080/00207179.2021.1961020

Published online: 06 Aug 2021.

INTERNATIONAL JOURNAL OF CONTROL
https://doi.org/10.1080/00207179.2021.1961020

Taylor & Francis
Taylor & Francis Group

Necessary conditions for partially observed optimal control of general McKean–Vlasov stochastic differential equations with jumps

Hakima Miloudi[a], Shahlar Meherrem [b,c], Imad Eddine Lakhdari[a] and Mokhtar Hafayed [a]

[a]Laboratory of Mathematical Analysis, Probability and Optimizations, Biskra University, Biskra, Algeria; [b]Department of Mathematics, Faculty of Science and Letters, Yasar University, Izmir, Turkey; [c]Institute of Control Systems, Azerbaijan National Academy of Sciences, Baku, Azerbaijan

ABSTRACT

In this paper, we establish necessary conditions of optimality for partially observed optimal control problems of Mckean–Vlasov type. The system is described by a controlled stochastic differential equation governed by Poisson random measure and an independent Brownian motion. The coefficients of the McKean–Vlasov system depend on the state of the solution process as well as of its probability law and the control variable. The proof of our result is based on Girsanov's theorem, variational equations and derivatives with respect to probability measure under convexity assumption. At the end of this paper, we apply our stochastic maximum principle to study partially observed linear quadratic control problem of McKean–Vlasov type with jumps and derive the explicit expression of the optimal control.

ARTICLE HISTORY
Received 26 December 2020
Accepted 22 July 2021

KEYWORDS
Partially observed optimal control; McKean–Vlasov stochastic system with jumps; probability measure; Girsanov's theorem; derivatives with respect to measure

MATHEMATICS SUBJECT CLASSIFICATIONS
93E20; 60H10

1. Introduction

Partially observed control problems have received much attention and became a powerful tool in many fields, such as mathematical finance, optimal control, and so on. From the viewpoint of reality, many situations, full information is not always available to controllers, but the partial one with noise, see, e.g. Djehiche and Tembine (2016), Fleming (1968), Lakhdari et al. (2021), Tang and Meng (2017), Wang and Wu (2009) and Wang et al. (2014) and the references therein. Necessary and sufficient conditions of optimality for system driven by Brownian motions and Poisson random measure where states and observations are correlated have been established by Xiao (2013). Partially observed optimal control problem for forward–backward stochastic systems with jump has been discussed by Wang et al. (2019). Stochastic maximum principle for partially observed forward–backward stochastic system with jumps and regime switching has been established by Zhang et al. (2018). Partially observed time-inconsistent stochastic linear-quadratic control problem with random jumps has been investigated by Wu and Zhuang (2018). The necessary conditions of optimality for forward–backward stochastic control systems with correlated state and observation noise have been obtained by Wang et al. (2013). A class of linear-quadratic optimal control problem of forward–backward stochastic differential equations with partial information has been studied by Wang et al. (2015). Recently, maximum principle for mean-field optimal stochastic control with partial-information has been discussed in Wang et al. (2014).

McKean–Vlasov stochastic differential equations (SDEs) are Itô's stochastic differential equations, where the coefficients of the state equation depend on the state of the solution process as well as of its probability law. This kind of equations was studied by Kac (1959) as a stochastic model for the Vlasov–Kinetic equation of plasma and the study of which was initiated by McKean (1966) to provide a rigorous treatment of special nonlinear partial differential equations. Optimal control problems for McKean–Vlasov SDEs has been investigated by many authors, for example, Buckdahn et al. (2016) proved the necessary conditions for general mean-field systems by using the tool of the second-order derivatives with respect to measures. Maximum principle for optimal control of McKean–Vlasov forward–backward stochastic differential equations (FBSDEs) with Lévy process via the differentiability with respect to probability law has been proved by Meherrem Hafayed (2019). Necessary and sufficient optimality conditions of optimal singular control problem for general Mckean–Vlasov differential equations have been discussed by Hafayed et al. (2018). A general necessary optimality conditions for stochastic continuous-singular control of McKean–Vlasov type equations, where the control domain is not assumed convex have been proved by Guenane et al. (2020). Stochastic maximum principle for partially observed optimal control problems of Mckean–Vlasov type has been established by Lakhdari et al. (2021).

In this paper, we prove a new stochastic maximum principle for a class of partially observed optimal control problems of Mckean–Vlasov type with jumps. The stochastic system under consideration is governed by a stochastic differential equation

CONTACT Mokhtar Hafayed hafa.mokh@yahoo.com Laboratory of Mathematical Analysis, Probability and Optimizations, Biskra University, P.O. Box 145, 07000 Biskra, Algeria

driven by Poisson random measure and an independent Brownian motion. The McKean–Vlasov SDEs with Poisson jump process is a type of stochastic process that has discrete movements, called jumps, with random arrival times, rather than continuous movement, typically modelled as a simple or compound Poisson process. The coefficients of our McKean–Vlasov dynamic depend nonlinearly on both the state process as well as of its probability law. The control domain is assumed to be convex. The derivatives with respect to probability measure and the associate It ô-formula are applied to prove our main results. Noting that the our general McKean–Vlasov partially observed control problem occur naturally in the probabilistic analysis of financial optimisation problems. Our class of partially observed control problem is strongly motivated by the recent study of the McKean–Vlasov games and recently play an important role in different fields of economics and finance (see, e.g., *conditional mean-variance portfolio selection problem with discrete movement in incomplete market*). As an illustration, by applying our maximum principle, McKean–Vlasov type linear quadratic control problem with jump is discussed, where the partially observed optimal control is obtained explicitly in feedback form.

The rest of the paper is organised as follows. Section 2 begins with a formulation of the partially observed control problem of general Mckean–Vlasov differential equations with jump processes. We give the notations and definitions of the derivatives with respect to probability measure and assumptions used throughout the paper. In Section 3, we prove the necessary conditions of optimality which are our main results. A linear quadratic control problem of this kind of partially observed control problem is also given in Section 4. At the end of this paper, some discussions with concluding remarks and future developments are presented in the last section.

2. Formulation of the problem and preliminaries

Let T is a fixed terminal time and $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ be a complete filtered probability space on which are defined two independent standard one-dimensional Brownian motions $W(\cdot)$ and $Y(\cdot)$. Let $\mathbb{R}^n$ is an n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n\times d$ matrices. Let $k(\cdot)$ be a stationary $\mathcal{F}_t$-Poisson point process with the characteristic measure $m(\mathrm{d}\theta)$. We denote by $N(\mathrm{d}\theta,\mathrm{d}t)$ the counting measure or Poisson measure induced by $k(\cdot)$, defined on $\Theta\times\mathbb{R}_+$, where Θ is a fixed nonempty subset of $\mathbb{R}$ with its Borel σ-field $\mathcal{B}(\Theta)$ and set $\widetilde{N}(\mathrm{d}\theta,\mathrm{d}t)=N(\mathrm{d}\theta,\mathrm{d}t)-m(\mathrm{d}\theta)\mathrm{d}t$ satisfying $\int_\Theta(1\wedge|\theta|^2)m(\mathrm{d}\theta)<\infty$ and $m(\Theta)<+\infty$. Let $\mathcal{F}_t^W$, $\mathcal{F}_t^Y$ and $\mathcal{F}_t^N$ be the natural filtration generated by $W(\cdot)$, $Y(\cdot)$ and $N(\cdot)$, respectively. We assume that

$$\mathcal{F}_t:=\mathcal{F}_t^W\vee\mathcal{F}_t^Y\vee\mathcal{F}_t^N\vee\mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets. We denote by $\langle\cdot,\cdot\rangle$ (resp. $|\cdot|$) the scalar product (resp., norm), $\mathbb{E}$ denotes the expectation on $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. Moreover, we denote by

(1) $L^2(r,s;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued deterministic function $\beta(\cdot)$, such that $\int_r^s|\beta(t)|^2\,\mathrm{d}t<+\infty$.
(2) $L^2(\mathcal{F}_t;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable ϕ, such that $\mathbb{E}|\phi|^2<+\infty$.
(3) $L^2_{\mathcal{F}}(r,s;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$, such that $\mathbb{E}\int_r^s|\psi(t)|^2\,\mathrm{d}t<+\infty$.
(4) $\mathbb{M}^2([0,T];\mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted measurable process $g(\cdot)$, such that

$$\mathbb{E}\int_0^T\int_\Theta\left|g(t,\theta)\right|^2m(\mathrm{d}\theta)\,\mathrm{d}t<+\infty.$$

(5) $\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ is the Hilbert space with inner product $(x,y)_2=\mathbb{E}[x\cdot y]$, $x,y\in\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ and the norm $\|x\|_2=\sqrt{(x,x)_2}$.
(6) $Q_2(\mathbb{R}^d)$ the space of all probability measures μ on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ with finite second moment, i.e. $\int_{\mathbb{R}^d}|x|^2\mu(\mathrm{d}x)<+\infty$, endowed with the following 2-Wasserstein metric; for $\mu,\nu\in Q_2(\mathbb{R}^d)$,

$$\mathbb{W}_2(\mu,\nu)=\inf\left\{\left[\int_{\mathbb{R}^d}|x-y|^2\rho\left(\mathrm{d}x,\mathrm{d}y\right)\right]^{\frac{1}{2}}:\rho\in Q_2\left(\mathbb{R}^{2d}\right),\ \rho\left(\cdot,\mathbb{R}^d\right)=\mu,\rho\left(\mathbb{R}^d,\cdot\right)=\nu\right\}.$$

Now, we recall briefly the main results of the differentiability with respect to probability measures was studied by Lions (2013) to derive our main result. The main idea is to identify a distribution $\mu\in Q_2(\mathbb{R}^d)$ with a random variables $\vartheta\in\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ so that $\mu=\mathbb{P}_\vartheta$. To be more precise, we assume that probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ is rich enough in the sense that for every $\mu\in Q_2(\mathbb{R}^d)$, there is a random variable $\vartheta\in\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ such that $\mu=\mathbb{P}_\vartheta$. It is well known that the probability space $([0,1],\mathcal{B}[0,1],\mathrm{d}x)$, where $\mathrm{d}x$ is the Borel measure has this property, see Buckdahn et al. (2016).

Definition 2.1 (Lift function): Let f be a given function such that $f:Q_2(\mathbb{R}^d)\to\mathbb{R}$. We define the lift function $\widetilde{f}:\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)\to\mathbb{R}$ such that

$$\widetilde{f}(Z):=f(\mathbb{P}_Z),\quad Z\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

Clearly, the lift function $\widetilde{f}$ of f, depends only on the law of $Z\in\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ and is independent of the choice of the representative Z.

Definition 2.2: A function $f:Q_2(\mathbb{R}^d)\to\mathbb{R}$ is said to be differentiable at $\mu_0\in Q_2(\mathbb{R}^d)$ if there exists $\vartheta_0\in\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ with $\mu_0=\mathbb{P}_{\vartheta_0}$ such that its lift function $\widetilde{f}$ is Fréchet differentiable at ϑ_0. More precisely, there exists a continuous linear functional $D\widetilde{f}(\vartheta_0):\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)\to\mathbb{R}$ such that

$$\begin{aligned}&\widetilde{f}(\vartheta_0+\xi)-\widetilde{f}(\vartheta_0)\\&\quad=\langle D\widetilde{f}(\vartheta_0),\xi\rangle+O(\|\xi\|_2)=D_\xi f(\mu_0)+O(\|\xi\|_2),\end{aligned}\tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the dual product on $\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$, and we will refer to $D_\xi f(\mu_0)$ as the Fréchet derivative of f at μ_0 in the direction ξ. In this case, we have

$$\begin{aligned}D_\xi f(\mu_0)&=\langle D\widetilde{f}(\vartheta_0),\xi\rangle\\&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f}(\vartheta_0+t\xi)\right|_{t=0},\quad\text{with }\mu_0=\mathbb{P}_{\vartheta_0}.\end{aligned}$$

By applying the Riesz' representation theorem, there is a unique random variable $z_0 \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ such that $\langle D\widetilde{f}(\vartheta_0),\xi\rangle = (z_0,\xi)_2 = \mathbb{E}[(z_0,\xi)_2]$, where $\xi \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$. It was shown, see the works of Buckdahn et al. (2016) and Lions (2013) that there exists a Boral function $h[\mu_0] : \mathbb{R}^d \to \mathbb{R}^d$, depending only on the law $\mu_0 = \mathbb{P}_{\vartheta_0}$ but not on the particular choice of the representative ϑ_0 such that $z_0 = h[\mu_0](\vartheta_0)$.

Thus, we can write (1) as

$$f\left(\mathbb{P}_{\vartheta}\right) - f\left(\mathbb{P}_{\vartheta_0}\right) = \left(h\left[\mu_0\right](\vartheta_0), \vartheta - \vartheta_0\right)_2 + O\left(\|\vartheta - \vartheta_0\|_2\right), \times \forall\, \vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d).$$

We denote

$$\partial_\mu f\left(\mathbb{P}_{\vartheta_0}, x\right) = h\left[\mu_0\right](x), \quad x \in \mathbb{R}^d.$$

Moreover, we have the following identities:

$$D\widetilde{f}\left(\vartheta_0\right) = z_0 = h\left[\mu_0\right]\left(\vartheta_0\right) = \partial_\mu f\left(\mathbb{P}_{\vartheta_0}, \vartheta_0\right)$$

and

$$D_\xi f\left(\mathbb{P}_{\vartheta_0}\right) = \left\langle \partial_\mu f\left(\mathbb{P}_{\vartheta_0}, \vartheta_0\right), \xi\right\rangle,$$

where $\xi = \vartheta - \vartheta_0$.

Remark 2.3: We note that for each $\mu \in Q_2(\mathbb{R}^d)$, $\partial_\mu f(\mathbb{P}_\vartheta,\cdot) = h[\mathbb{P}_\vartheta](\cdot)$ is only defined in a $\mathbb{P}_\vartheta(\mathrm{d}x)$-*a.e* sense, where $\mu = \mathbb{P}_\vartheta$.

Definition 2.4 (Space of differentiable functions in $Q_2(\mathbb{R}^d)$): We say that the function $f \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^d))$ if for all $\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$, there exists a $\mathbb{P}_\vartheta$-modification of $\partial_\mu f(\mathbb{P}_\vartheta,\cdot)$ such that $\partial_\mu f : Q_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipchitz continuous. That is for some $C > 0$, it holds that

(i) $|\partial_\mu f(\mu,x)| \le C, \forall \mu \in Q_2(\mathbb{R}^d),\ \forall\, x \in \mathbb{R}^d$;
(ii) $|\partial_\mu f(\mu_1,x_1) - \partial_\mu f(\mu_2,x_2)| \le C(\mathbb{W}_2(\mu_1,\mu_2) + |x_1 - x_2|)$, $\forall \mu_1,\mu_2 \in Q_2(\mathbb{R}^d), \forall x_1,x_2 \in \mathbb{R}^d$.

We would like to point out that the version of $\partial_\mu f(\mathbb{P}_\vartheta,\cdot)$, $\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ indicated in the above definition is unique (see Remark 2.2 in Buckdahn et al. (2016) for more information).

Let $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathcal{F}}_t,\widehat{\mathbb{P}})$ be a copy of the probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. For any pair of random variable $(\vartheta,\xi) \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d) \times \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$, we let $(\widehat{\vartheta},\widehat{\xi})$ be an independent copy of (ϑ,ξ) defined on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathcal{F}}_t,\widehat{\mathbb{P}})$. We consider the product probability space $(\Omega \times \widehat{\Omega}, \mathcal{F} \otimes \widehat{\mathcal{F}}, \mathcal{F}_t \otimes \widehat{\mathcal{F}}_t, \mathbb{P}\otimes\widehat{\mathbb{P}})$ and setting $(\widehat{\vartheta},\widehat{\xi})(w,\widehat{w}) = (\vartheta(\widehat{w}),\xi(\widehat{w}))$ for any $(w,\widehat{w}) \in \Omega \times \widehat{\Omega}$. Let $(\widehat{u}(t),\widehat{x}(t))$ be an independent copy of $(u(t),x(t))$ so that $\mathbb{P}_{x(t)} = \widehat{\mathbb{P}}_{\widehat{x}(t)}$. We denote by $\widehat{\mathbb{E}}$ the expectation under probability measure $\widehat{\mathbb{P}}$ and $\mathbb{P}_X = \mathbb{P}\circ X^{-1}$ denotes the law of the random variable X.

Let U be a nonempty convex subset of $\mathbb{R}^k$. An admissible control v is an $\mathcal{F}_t^Y$-adapted process with values in U satisfies $\sup_{t\in[0,T]} \mathbb{E}|v_t|^n < \infty$, $n = 2,3,\ldots$. We denote by $\mathcal{U}_{ad}([0,T])$ the set of the admissible control variables.

For given control process $v(\cdot) \in \mathcal{U}_{ad}([0,T])$, the dynamics of the controlled system take the following type:

$$\begin{cases} \mathrm{d}x^v(t) = f\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right)\mathrm{d}t \\ \qquad + \sigma\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right)\mathrm{d}W(t) \\ \qquad + c\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right)\mathrm{d}\widetilde{W}(t) \\ \qquad + \int_\Theta g\left(t, x^v(t_-), \mathbb{P}_{x^v(t_-)}, v(t), \theta\right)\widetilde{N}(\mathrm{d}\theta, \mathrm{d}t), \\ x^v(0) = x_0,\ t \in [0,T], \end{cases} \tag{2}$$

where $\mathbb{P}_X = \mathbb{P}\circ X^{-1}$ denotes the law of the random variable X. The coefficients

$$f : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^d) \times U \to \mathbb{R}^n,\ \sigma : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^d) \times U \to \mathbb{R}^{n\times d},\ c : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^d) \times U \to \mathbb{R}^{n\times d},\ g : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^d) \times U \times \Theta \to \mathbb{R}^{n\times d}$$

are given deterministic functions.

Suppose that the state processes $x^v(\cdot)$ cannot be observed directly, but the controllers can observe a related noisy process $Y(\cdot)$, which is governed by the following equation:

$$\begin{cases} \mathrm{d}Y(t) = h(t, x^v(t), v(t))\,\mathrm{d}t + \mathrm{d}\widetilde{W}(t) \\ Y(0) = 0, \end{cases} \tag{3}$$

where $h : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^r$, and $\widetilde{W}(\cdot)$ is a stochastic process depending on the control $v(\cdot)$.

Remark 2.5: Note that if the diffusion term $c \neq 0$ in Equation (2), then there exist the correlated noise $\widetilde{W}(\cdot)$ between the state and observation.

Consider the cost functional

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t))\,\mathrm{d}t + \psi(x^v(T), \mathbb{P}_{x^v(T)})\right]. \tag{4}$$

Here, $l : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $\psi : \mathbb{R}^n \times Q_2(\mathbb{R}) \to \mathbb{R}$ and $\mathbb{E}^v$ stands for the mathematical expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^v)$.

In this paper, we shall make use of the following standing assumption.

Assumption (H1): The maps $f,\sigma,c,l : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$ and $\psi : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$ are measurable in all variables. Moreover, $f(t,\cdot,\cdot,v)$, $\sigma(t,\cdot,\cdot,v)$, $c(t,\cdot,\cdot,v), l(t,\cdot,\cdot,v)$, $g(t,\cdot,\cdot,v,\theta) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}),\mathbb{R})$ and $\psi(\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}),\mathbb{R})$ for all $v \in U$.

Assumption (H2): Denoting $\varphi(x,\mu) = f(t,x,\mu,v), \sigma(t,x,\mu,v)$, $c(t,x,\mu,v)$, $l(t,x,\mu,v)$, $g(t,x,\mu,v,\theta)$, $\psi(x,\mu)$, the function $\varphi(\cdot,\cdot)$ satisfies the following properties.

(i) For fixed $x \in \mathbb{R}$ and $\mu \in Q_2(\mathbb{R})$, the function $\varphi(\cdot,\mu) \in \mathbb{C}_b^1(\mathbb{R})$ and $\varphi(x,\cdot) \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^d),\mathbb{R})$.
(ii) All the derivatives φ_x and $\partial_\mu\varphi$, for $\varphi = f,\sigma,c,l,\psi$ are bounded and Lipschitz continuous, with Lipschitz constants independent of $v \in U$. Moreover, there exists a

constants $C(T, m(\Theta)) > 0$ independent to v and Θ such that

$$\sup_{\theta\in\Theta}\left|\partial_x g\left(t,x,\mu,u,\theta\right)\right| + \sup_{\theta\in\Theta}\left|\partial_\mu g\left(t,x,\mu,u,\theta\right)\right| \le C.$$

$$\begin{aligned}&\sup_{\theta\in\Theta}\left|g_x\left(t,x,\mu,u,\theta\right) - g_x\left(t,x',\mu',u,\theta\right)\right|\\ &\quad+\sup_{\theta\in\Theta}\left|\partial_\mu g\left(t,x,\mu,u,\theta\right) - \partial_\mu g\left(t,x',\mu',u,\theta\right)\right|\\ &\le C\left[\left|x-x'\right| + \mathbb{W}_2(\mu,\mu')\right]\end{aligned}$$

(iii) The functions f, σ, c, g and l are continuously differentiable with respect to control variable v, and all their derivatives are continuous and bounded. Moreover, there exists a constants $C = C(T, m(\Theta)) > 0$ such that

$$\sup_{\theta\in\Theta}\left|g_u\left(t,x,\mu,u,\theta\right)\right| \le C.$$

(iv) The function h is continuously differentiable in x and continuous in v, its derivatives and h are all uniformly bounded.

Clearly, under Assumptions (H1) and (H2), for any $v(\cdot) \in \mathcal{U}_{ad}([0,T])$ the McKean–Vlasov SDE-(2) admits a unique strong solution $x^v(t)$ given by

$$\begin{aligned}x^v\left(t\right) = x_0 &+ \int_0^t f\left(s, x^v\left(s\right), \mathbb{P}_{x^v(s)}, v\left(s\right)\right)\mathrm{d}s\\ &+\sigma\left(s, x^v\left(s\right), \mathbb{P}_{x^v(s)}, v\left(s\right)\right)\mathrm{d}W\left(s\right)\\ &+c\left(s, x^v\left(s\right), \mathbb{P}_{x^v(s)}, v\left(s\right)\right)\mathrm{d}\widetilde{W}\left(s\right)\\ &+\int_0^t\int_\Theta g\left(s, x^v\left(s_-\right), \mathbb{P}_{x^v(s_-)}, v\left(s\right),\theta\right)\widetilde{N}\left(\mathrm{d}\theta, \mathrm{d}s\right).\end{aligned}$$

We define $\mathrm{d}\mathbb{P}^v = \rho^v(t)\,\mathrm{d}\mathbb{P}$ with

$$\rho^v(t) = \exp\left\{\int_0^t h(s, x^v(s), v(s))\,\mathrm{d}Y(s) - \frac{1}{2}\int_0^t \left|h(s, x^v(s), v(s))\right|^2\mathrm{d}s\right\},$$

where $\rho^v(\cdot)$ is the unique $\mathcal{F}_t^Y$-adapted solution of the linear stochastic differential equation

$$\begin{cases}\mathrm{d}\rho^v(t) = \rho^v(t)h\left(t, x^v(t), v(t)\right)\mathrm{d}Y(t),\\ \rho^v(0) = 1.\end{cases} \tag{5}$$

By virtue of Itô's formula, we can prove that $\sup_{t\in[0,T]}\mathbb{E}|\rho_t^v|^m < \infty$, $m = 2, 3, \ldots$. Hence, by Girsanov's theorem and Assumptions (H1) and (H2), $\mathbb{P}^v$ is a new probability measure and $(W(\cdot), \widetilde{W}(\cdot))$ is two-dimensional standard Brownian motion defined on the new probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^v)$.

Our partially observed optimal control problem becomes the following minimisation problem: to minimise the cost functional in (4) over $v(\cdot) \in \mathcal{U}_{ad}([0,T])$ subject to Equations (2) – (3), such that

$$J(u(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{ad}([0,T])} J\left(v\left(\cdot\right)\right). \tag{6}$$

Obviously, we can rewritten the cost functional (4) as

$$\begin{aligned}J(v(\cdot)) = \mathbb{E}\Big[\int_0^T &\rho^v(t)l(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t))\,\mathrm{d}t\\ &+\rho^v(T)\psi(x^v(T), \mathbb{P}_{x^v(T)})\Big].\end{aligned} \tag{7}$$

So the original optimisation problem is equivalent to minimising
(7) over $v(\cdot) \in \mathcal{U}_{ad}([0,T])$, subject to (2)–(5).

The main purpose of this paper is to prove stochastic maximum principle, also called necessary optimality conditions for the partially observed optimal control of Mckean–Vlasov SDE with jumps.

3. Necessary conditions of optimality

In this section, we prove the necessary conditions of optimality for our partially observed optimal control problem of general Mckean–Vlasov stochastic differential equations with jumps. The proof is based on Girsanov's theorem, the derivatives with respect to probability measure and on introducing the variational equations with some estimates of their solutions.

Hamiltonian. We define the Hamiltonian

$$H : [0,T]\times\mathbb{R}\times Q_2\left(\mathbb{R}\right)\times U\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},$$

associated with our control problem by

$$\begin{aligned}&H(t, x, \mu, v, \Phi, Q, \overline{Q}, K, R)\\ &= l(t, x, \mu, v) + f(t, x, \mu, v)\Phi + \sigma(t, x, \mu, v)Q\\ &\quad+ c(t, x, \mu, v)\overline{Q} + h\left(t, x, v\right)K\\ &\quad+\int_\Theta g\left(t, x, \mu, v, \theta\right)R\left(\theta\right)m\left(\mathrm{d}\theta\right).\end{aligned} \tag{8}$$

Let $(u(\cdot), x(\cdot))$ be the optimal solution of the control problem (2)–(6). Then for any $0 \le \varepsilon \le 1$ and $v(\cdot) \in \mathcal{U}_{ad}([0,T])$, we define the variational control by $v^\varepsilon(\cdot) = u(\cdot) + \varepsilon v(\cdot) \in \mathcal{U}_{ad}([0,T])$. We denote by $x^\varepsilon(\cdot), x(\cdot), \rho^\varepsilon(\cdot), \rho(\cdot)$ the state trajectories of (2) and (5) corresponding respectively to $v^\varepsilon(\cdot)$ and $u(\cdot)$.

For simplification, we introduce the short-hand notation

$$\begin{aligned}\varphi\left(t\right) &= \varphi\left(t, x(t), \mathbb{P}_{x(t)}, u(t)\right),\\ \varphi^\varepsilon\left(t\right) &= \varphi(t, x^\varepsilon(t), \mathbb{P}_{x^\varepsilon(t)}, v^\varepsilon(t)),\end{aligned}$$

and

$$\begin{aligned}&g\left(t,\theta\right) = g(t, x(t_-), \mathbb{P}_{x(t_-)}, u(t), \theta), \quad h\left(t\right) = h\left(t, x(t), u(t)\right),\\ &g^\varepsilon\left(t,\theta\right) = g(t, x^\varepsilon(t_-), \mathbb{P}_{x^\varepsilon(t_-)}, v^\varepsilon(t), \theta), \quad h^\varepsilon\left(t\right) = h\left(t, x^\varepsilon(t), v^\varepsilon(t)\right),\end{aligned}$$

where g, h and $\varphi = f, \sigma, c, l$ as well as their partial derivatives with respect to x and v.

Also, we will denote for $\varphi = f, \sigma, c, l$ and g:

$$\partial_\mu \varphi(t) = \partial_\mu \varphi\left(t, x(t), \mathbb{P}_{x(t)}, u(t); \widehat{x}(t)\right),$$
$$\partial_\mu \widehat{\varphi}(t) = \partial_\mu \varphi\left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t); x(t)\right),$$

and

$$\partial_\mu g(t,\theta) = \partial_\mu g\left(t, x(t_-), \mathbb{P}_{x(t_-)}, u(t), \theta; \widehat{x}(t)\right),$$
$$\partial_\mu \widehat{g}(t,\theta) = \partial_\mu g\left(t, \widehat{x}(t), \mathbb{P}_{x(t)}, \widehat{u}(t), \theta; x(t)\right).$$

Now, we introduce the following variational equations:

$$\begin{cases} d\phi(t) = \left[f_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\phi}(t)\right] + f_v(t)v(t)\right]dt \\ \qquad + \left[\sigma_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \sigma(t)\widehat{\phi}(t)\right] + \sigma_v(t)v(t)\right]dW(t) \\ \qquad + \left[c_x(t)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu c(t)\widehat{\phi}(t)\right] + c_v(t)v(t)\right]d\widetilde{W}(t) \\ \qquad + \int_\Theta \left[g_x(t,\theta)\phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta)\widehat{\phi}(t)\right] + g_v(t,\theta)v(t)\right] \\ \qquad \widetilde{N}(d\theta, dt), \\ \phi(0) = 0, \end{cases} \tag{9}$$

and

$$\begin{cases} d\rho_1(t) = \left[\rho_1(t)h(t) + \rho(t)h_x(t)\phi(t) + \rho(t)h_v(t)v(t)\right]dY(t), \\ \rho_1(0) = 0. \end{cases} \tag{10}$$

Under Assumptions (H1) and (H2), Équations (9) and (10) admit a unique adapted solutions $\phi(\cdot)$ and $\rho_1(\cdot)$, respectively.

Adjoint equation. We are now ready to introduce two new adjoint equations that will be the building blocks of the stochastic maximum principle.

$$\begin{cases} -dy(t) = l(t)\,dt - z(t)\,dW(t) - K(t)\,d\widetilde{W}(t) \\ \qquad\qquad - \int_\Theta R(t,\theta)\widetilde{N}(d\theta, dt), \\ y(T) = \psi(x(T), \mathbb{P}_{x(T)}), \end{cases} \tag{11}$$

and

$$\begin{cases} -d\Phi(t) = \left[f_x(t)\Phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}(t)\widehat{\Phi}(t)\right]\right. \\ \qquad + \sigma_x(t)Q(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\sigma}(t)\widehat{Q}(t)\right] \\ \qquad + c_x(t)\overline{Q}(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{c}(t)\widehat{\overline{Q}}(t)\right] \\ \qquad + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{l}(t)\right] \\ \qquad + \int_\Theta \left[g_x(t,\theta)R(t,\theta) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(t,\theta)\widehat{R}(t,\theta)\right]\right] \\ \qquad \left.\times\, m(d\theta) + h_x(t)K(t)\right]dt \\ \qquad - Q(t)\,dW(t) - \overline{Q}(t)\,d\widetilde{W}(t) \\ \qquad - \int_\Theta R(t,\theta)\widetilde{N}(d\theta, dt), \\ \Phi(T) = \psi_x(x(T), \mathbb{P}_{x(T)}) + \widehat{\mathbb{E}}\left[\partial_\mu \psi(\widehat{x}(T), \mathbb{P}_{x(T)}; x(T))\right]. \end{cases} \tag{12}$$

Clearly, under Assumption (H1) and (H2), it is easy to prove that BSDEs (11) and (12) admits a unique strong solution, given by

$$y(t) = \psi(x(T), \mathbb{P}_{x(T)}) - \int_t^T l(s)\,ds + \int_t^T z(s)\,dW(s)$$
$$+ \int_t^T K(s)\,d\widetilde{W}(s) + \int_t^T \int_\Theta R(s,\theta)\widetilde{N}(d\theta, ds),$$

and

$$\Phi(t) = \psi_x(x(T), \mathbb{P}_{x(T)}) + \widehat{\mathbb{E}}\left[\partial_\mu \psi(\widehat{x}(T), \mathbb{P}_{x(T)}; x(T))\right].$$
$$- \int_t^T \left[f_x(s)\Phi(s) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}(s)\widehat{\Phi}(s)\right] + \sigma_x(s)Q(s)\right.$$
$$+ \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\sigma}(s)\widehat{Q}(s)\right]$$
$$+ c_x(s)\overline{Q}(s) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{c}(s)\widehat{\overline{Q}}(s)\right] + l_x(s) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{l}(s)\right]$$
$$+ \int_\Theta \left[g_x(s,\theta)R(s,\theta) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(s,\theta)\widehat{R}(s,\theta)\right]\right] m(d\theta)$$
$$\left. + h_x(s)K(s)\right] ds$$
$$+ \int_t^T Q(s)\,dW(s) + \int_t^T \overline{Q}(s)\,d\widetilde{W}(s)$$
$$+ \int_t^T \int_\Theta R(s,\theta)\widetilde{N}(d\theta, ds),$$

The main result of this paper is stated in the following theorem.

Theorem 3.1: *Let Assumptions* (H1) *and* (H2) *hold. Let* $(u(\cdot), x(\cdot))$ *be the optimal solution of the control problem* (2)–(6). *Then there exists* $(\Phi(\cdot), Q(\cdot), \overline{Q}(\cdot), K(\cdot), R(\cdot,\theta))$ *solution of* (12), *such that for any* $v \in U$, *we have*

$$\mathbb{E}^u\left[H_v(t, x(t), \mathbb{P}_{x(t)}, u(t), \Phi(t), Q(t), \overline{Q}(t), K(t),\right.$$
$$\left. R(t,\theta))(v(t) - u(t)) \mid \mathcal{F}_t^Y\right] \geq 0, \quad \text{a.s., a.e.,}$$

where the Hamiltonian function H is defined by (8).

In order to prove our main result in Theorem 3.1, we present some auxiliary results.

Lemma 3.2: *Suppose that Assumptions* (H1) *and* (H2) *hold. Then, we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left|x^\varepsilon(t) - x(t)\right|^2\right] = 0.$$

Proof: Applying standard estimates, the Burkholder–Davis–Gundy inequality, and Proposition A.1 (Appendix) we have

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left|x^\varepsilon(t) - x(t)\right|^2\right]$$
$$\leq \mathbb{E}\int_0^t \left|f^\varepsilon(s) - f(s)\right|^2 ds + \mathbb{E}\int_0^t \left|\sigma^\varepsilon(s) - \sigma(s)\right|^2 ds$$
$$+ \mathbb{E}\int_0^t \left|c^\varepsilon(s) - c(s)\right|^2 ds$$
$$+ \mathbb{E}\int_0^t \int_\Theta \left|g^\varepsilon(s,\theta) - g(s,\theta)\right|^2 m(d\theta)\,ds.$$

According to the Lipschitz conditions on the coefficients f, σ, c and g with respect to x, μ and u, (Assumption (H2)-(ii)), we get

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left|x^\varepsilon(t) - x(t)\right|^2\right]$$

$$\leq C_T\mathbb{E}\int_0^t\left[\left|x^\varepsilon(s)-x(s)\right|^2+\left|\mathbb{W}_2\left(\mathbb{P}_{x^\varepsilon(s)},\mathbb{P}_{x(s)}\right)\right|^2\right]\mathrm{d}s$$

$$+C_T\varepsilon^2\mathbb{E}\int_0^t|v(s)|^2\ \mathrm{d}s. \tag{13}$$

From the definition of Wasserstein metric $\mathbb{W}_2(\cdot,\cdot)$, we have

$$\begin{aligned}\mathbb{W}_2&\left(\mathbb{P}_{x^\varepsilon(s)},\mathbb{P}_{x(s)}\right)\\ &=\inf\left\{\left[\mathbb{E}\left|\widetilde{x}^\varepsilon(s)-\widetilde{x}(s)\right|^2\right]^{\frac{1}{2}},\right.\\ &\quad\text{for all }\widetilde{x}^\varepsilon(\cdot),\widetilde{x}(\cdot)\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right),\\ &\quad\left.\text{with }\mathbb{P}_{x^\varepsilon(s)}=\mathbb{P}_{\widetilde{x}^\varepsilon(s)}\text{ and }\mathbb{P}_{x(s)}=\mathbb{P}_{\widetilde{x}(s)}\right\}\\ &\leq\left[\mathbb{E}\left|x^\varepsilon(s)-x(s)\right|^2\right]^{\frac{1}{2}}.\end{aligned}\tag{14}$$

By Definition 2.2 and from (13) and (14), we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x^\varepsilon(t)-x(t)\right|^2\right]$$

$$\leq C_T\mathbb{E}\int_0^t\sup_{r\in[0,s]}\left|x^\varepsilon(r)-x(r)\right|^2\ \mathrm{d}s+M_T\varepsilon^2.$$

By applying Gronwall's inequality, the desired result follows immediately by letting ε go to zero. ■

Lemma 3.3: *Suppose that Assumptions* (H1) *and* (H2) *hold. Then, we have*

$$\lim_{\varepsilon\to0}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{x^\varepsilon(t)-x(t)}{\varepsilon}-\phi(t)\right|^2=0. \tag{15}$$

Proof: We put

$$\eta^\varepsilon(t)=\frac{x^\varepsilon(t)-x(t)}{\varepsilon}-\phi(t),\quad t\in[0,T].$$

To simplify, we will use the following notations, for $\varphi=f,\sigma,c,l$ and g:

$$\begin{aligned}\varphi_x^{\lambda,\varepsilon}(t)&=\varphi_x\left(t,x^{\lambda,\varepsilon}(t),\mathbb{P}_{x^\varepsilon(t)},v^\varepsilon(t)\right),\\ g_x^{\lambda,\varepsilon}(t,\theta)&=g_x\left(t,x^{\lambda,\varepsilon}(t),\mathbb{P}_{x^\varepsilon(t)},v^\varepsilon(t),\theta\right),\\ \partial_\mu^{\lambda,\varepsilon}\varphi(t)&=\partial_\mu\varphi(s,x^\varepsilon(t),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(t)},v^\varepsilon(t);\widehat{x}(t)),\\ \partial_\mu^{\lambda,\varepsilon}g(t,\theta)&=\partial_\mu g(t,x^\varepsilon(t),\mathbb{P}_{\widehat{x}^{\lambda,\varepsilon}(t)},v^\varepsilon(t),\theta;\widehat{x}(t)),\end{aligned}$$

and

$$\begin{aligned}x^{\lambda,\varepsilon}(s)&=x(s)+\lambda\varepsilon\left(\eta^\varepsilon(s)+\phi(s)\right),\\ \widehat{x}^{\lambda,\varepsilon}(s)&=x(s)+\lambda\varepsilon(\widehat{\eta}^\varepsilon(s)+\widehat{\phi}(s)),\\ v^{\lambda,\varepsilon}(s)&=u(s)+\lambda\varepsilon v(s).\end{aligned}$$

Since $D_\xi f(\mu_0)=\langle D\widetilde{f}(\vartheta_0),\xi\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f}(\vartheta_0+t\xi)|_{t=0}$, we have the following form of the Taylor expansion:

$$f\left(\mathbb{P}_{\vartheta_0+\xi}\right)-f\left(\mathbb{P}_{\vartheta_0}\right)=D_\xi f\left(\mathbb{P}_{\vartheta_0}\right)+\mathcal{R}(\xi),$$

where $\mathcal{R}(\xi)$ is of order $O(\|\xi\|_2)$ with $O(\|\xi\|_2)\to0$ for $\xi\in\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$.

$$\begin{aligned}\eta^\varepsilon(t)&=\frac{1}{\varepsilon}\int_0^t\left[f^\varepsilon(s)-f(s)\right]\mathrm{d}s+\frac{1}{\varepsilon}\int_0^t\left[\sigma^\varepsilon(s)-\sigma(s)\right]\mathrm{d}W(s)\\ &+\frac{1}{\varepsilon}\int_0^t\left[c^\varepsilon(s)-c(s)\right]\mathrm{d}\widetilde{W}(s)\\ &+\frac{1}{\varepsilon}\int_0^t\int_\Theta\left[g^\varepsilon(s,\theta)-g(s,\theta)\right]\widetilde{N}(\mathrm{d}\theta,\mathrm{d}s)\\ &-\int_0^t\left[f_x(s)\phi(s)+\widehat{\mathbb{E}}\left[\partial_\mu f(s)\widehat{\phi}(s)\right]+f_v(s)v(s)\right]\mathrm{d}s\\ &-\int_0^t\left[\sigma_x(s)\phi(s)+\widehat{\mathbb{E}}\left[\partial_\mu\sigma(s)\widehat{\phi}(s)\right]+\sigma_v(s)v(s)\right]\mathrm{d}W(s)\\ &-\int_0^t\left[c_x(s)\phi(s)+\widehat{\mathbb{E}}\left[\partial_\mu c(s)\widehat{\phi}(s)\right]+c_v(s)v(s)\right]\mathrm{d}\widetilde{W}(s)\\ &-\int_0^t\int_\Theta\left[g_x(s,\theta)\phi(s)+\widehat{\mathbb{E}}\left[\partial_\mu g(s,\theta)\widehat{\phi}(s)\right]\right.\\ &\left.+g_v(s,\theta)v(s)\right]\widetilde{N}(\mathrm{d}\theta,\mathrm{d}s).\end{aligned}$$

We decompose $\frac{1}{\varepsilon}\int_0^t[f^\varepsilon(s)-f(s)]\,\mathrm{d}s$ into the following three parts:

$$\begin{aligned}\frac{1}{\varepsilon}&\int_0^t\left[f^\varepsilon(s)-f(s)\right]\mathrm{d}s\\ &=\frac{1}{\varepsilon}\int_0^t\left[f^\varepsilon(s)-f(s,x(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s))\right]\mathrm{d}s\\ &+\frac{1}{\varepsilon}\int_0^t\left[f(s,x(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s))-f(s,x(s),\mathbb{P}_{x(s)},v^\varepsilon(s))\right]\mathrm{d}s\\ &+\frac{1}{\varepsilon}\int_0^t\left[f(s,x(s),\mathbb{P}_{x(s)},v^\varepsilon(s))-f(s)\right]\mathrm{d}s.\end{aligned}$$

We notice that

$$\begin{aligned}\frac{1}{\varepsilon}&\int_0^t\left[f^\varepsilon(s)-f(s,x(s),\mathbb{P}_{x^\varepsilon(s)},v^\varepsilon(s))\right]\mathrm{d}s\\ &=\int_0^t\int_0^1\left[f_x^{\lambda,\varepsilon}(s)\left(\eta^\varepsilon(s)+\phi(s)\right)\right]\mathrm{d}\lambda\,\mathrm{d}s,\\ \frac{1}{\varepsilon}&\int_0^t\left[f^\varepsilon(s)-f(s,x^\varepsilon(s),\mathbb{P}_{x(s)},v^\varepsilon(s)\right]\mathrm{d}s\\ &=\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\partial_\mu^{\lambda,\varepsilon}f(s)\left(\widehat{\eta}^\varepsilon(s)+\widehat{\phi}(s)\right)\right]\mathrm{d}\lambda\,\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varepsilon}&\int_0^t\left[f\left(s,x(s),\mathbb{P}_{x(s)},v^\varepsilon(s)\right)-f(s)\right]\mathrm{d}s\\ &=\int_0^t\int_0^1\left[f_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\right)v(s)\right]\mathrm{d}\lambda\,\mathrm{d}s.\end{aligned}$$

The analogue relations hold for σ, c and g. Therefore, we get

$$\begin{aligned}\mathbb{E}&\left[\sup_{s\in[0,t]}\left|\eta^\varepsilon(s)\right|^2\right]\\ &=C(t)\,\mathbb{E}\left[\int_0^t\int_0^1\left|f_x^{\lambda,\varepsilon}(s)\,\eta^\varepsilon(s)\right|^2\ \mathrm{d}\lambda\,\mathrm{d}s\right.\\ &+\int_0^t\int_0^1\widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon}f(s)\,\widehat{\eta}^\varepsilon(s)\right|^2\ \mathrm{d}\lambda\,\mathrm{d}s\end{aligned}$$

$$+\int_0^t\int_0^1\left|\sigma_x^{\lambda,\varepsilon}(s)\,\eta^\varepsilon(s)\right|^2\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon}\sigma(s)\widehat{\eta}^\varepsilon(s)\right|^2\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\left|c_x^{\lambda,\varepsilon}(s)\,\eta^\varepsilon(s)\right|^2\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon}c(s)\widehat{\eta}^\varepsilon(s))\right|^2\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_\Theta\int_0^1\left|g_x^{\lambda,\varepsilon}(s,\theta)\,\eta^\varepsilon(s)\right|^2\mathrm{d}\lambda m(\mathrm{d}\theta)\,\mathrm{d}s$$
$$\left.+\int_0^t\int_\Theta\int_0^1\widehat{\mathbb{E}}\left|\partial_\mu^{\lambda,\varepsilon}g(s,\theta)\,\widehat{\eta}^\varepsilon(s))\right|^2\mathrm{d}\lambda m(\mathrm{d}\theta)\,\mathrm{d}s\right]$$
$$+C(t)\,\mathbb{E}\left[\sup_{s\in[0,t]}\left|\gamma^\varepsilon(s)\right|^2\right],$$

where

$$\gamma^\varepsilon(t)=\int_0^t\int_0^1\left[f_x^{\lambda,\varepsilon}(s)-f_x(s)\right]\phi(s)\,\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}f(s)-\partial_\mu f(s)\right)\widehat{\phi}(s)\right]\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\left[f_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\right)-f_v(s)\right]v(s)\,\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\left[\sigma_x^{\lambda,\varepsilon}(s)-\sigma_x(s)\right]\phi(s)\,\mathrm{d}\lambda\,\mathrm{d}W(s)$$
$$+\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}\sigma(s)-\partial_\mu\sigma(s)\right)\widehat{\phi}(s)\right]\mathrm{d}\lambda\,\mathrm{d}W(s)$$
$$+\int_0^t\int_0^1\left[\sigma_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\right)-\sigma_v(s)\right]$$
$$\times\, v(s)\,\mathrm{d}\lambda\,\mathrm{d}W(s)$$

$$+\int_0^t\int_0^1\left[c_x^{\lambda,\varepsilon}(s)-c_x(s)\right]\phi(s)\,\mathrm{d}\lambda\,\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}c(s)-\partial_\mu c(s)\right)\widehat{\phi}(s)\right]\mathrm{d}\lambda\,\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_0^1\left[c_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s)\right)-c_v(s)\right]v(s)\,\mathrm{d}\lambda\,\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_\Theta\int_0^1\left[g_x^{\lambda,\varepsilon}(s,\theta)-g_x(s,\theta)\right]\phi(s_-)\,\mathrm{d}\lambda\widetilde{N}(\mathrm{d}\theta,\mathrm{d}s)$$
$$+\int_0^t\int_\Theta\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}g(s,\theta)-\partial_\mu g(s,\theta)\right)\widehat{\phi}(s_-)\right]\mathrm{d}\lambda\widetilde{N}(\mathrm{d}\theta,\mathrm{d}s)$$
$$+\int_0^t\int_\Theta\int_0^1\left[g_v\left(s,x(s),\mathbb{P}_{x(s)},v^{\lambda,\varepsilon}(s),\theta\right)-g_v(s,\theta)\right]$$
$$\times\, v(s)\,\mathrm{d}\lambda\widetilde{N}(\mathrm{d}\theta,\mathrm{d}s).$$

Now, the derivatives of f,σ,c and g with respect to (x,μ,v) are Lipschitz continuous in (x,μ,v), we get

$$\lim_{\varepsilon\to0}\mathbb{E}\left[\sup_{s\in[0,T]}\left|\gamma^\varepsilon(s)\right|^2\right]=0.$$

Since the derivatives of f,σ,c and γ are bounded with respect to (x,μ,v), we have

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left|\eta^\varepsilon(s)\right|^2\right]$$
$$\le C(t)\left\{\mathbb{E}\int_0^t\left|\eta^\varepsilon(s)\right|^2\mathrm{d}s+\mathbb{E}\left[\sup_{s\in[0,t]}\left|\gamma^\varepsilon(s)\right|^2\right]\right\}.$$

From Gronwall's lemma, we obtain $\forall t\in[0,T]$

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left|\eta^\varepsilon(s)\right|^2\right]$$
$$\le C(t)\left\{\mathbb{E}\left[\sup_{s\in[0,t]}\left|\gamma^\varepsilon(s)\right|^2\right]\exp\left\{\int_0^t C(s)\,\mathrm{d}s\right\}\right\}.$$

Finally, putting $t=T$ and letting ε go to zero, the proof of Lemma 3.3 is complete. ∎

Now, we introduce the following lemma which play an important role in computing the variational inequality for the cost functional (7) subject to (2) and (5).

Lemma 3.4: *Let Assumption* (H1) *hold. Then, we have*

$$\lim_{\varepsilon\to0}\sup_{0\le t\le T}\mathbb{E}\left|\frac{\rho^\varepsilon(t)-\rho(t)}{\varepsilon}-\rho_1(t)\right|^2=0. \tag{16}$$

Proof: From the definition of $\rho(\cdot)$ and $\rho_1(\cdot)$, we obtain

$$\rho(t)+\varepsilon\rho_1(t)=1+\int_0^t\rho(s)h(s)\,\mathrm{d}Y(s)$$
$$+\varepsilon\int_0^t\left[\rho_1(s)\,h(s)+\rho(s)h_x(s)\,\phi(s)\right.$$
$$\left.+\rho(s)h_v(s)\,v(s)\right]\mathrm{d}Y(s)$$
$$=1+\varepsilon\int_0^t\rho_1(s)h(s)\,\mathrm{d}Y(s)+\int_0^t\rho(s)\,h(s,x(s)$$
$$+\varepsilon\phi(s),u(s)+\varepsilon v(s))\mathrm{d}Y(s)$$
$$-\varepsilon\int_0^t\rho(s)\left[A^\varepsilon(s)\right]\mathrm{d}Y(s),$$

where

$$A^\varepsilon(s)=\int_0^1\left[h_x(s,x(s)+\lambda\varepsilon\phi(s),u(s)\right.$$
$$\left.+\lambda\varepsilon v(s))-h_x(s)\right]\mathrm{d}\lambda\phi(s)$$
$$+\int_0^1\left[h_v(s,x(s)+\lambda\varepsilon\phi(s),u(s)\right.$$
$$\left.+\lambda\varepsilon v(s))-h_v(s)\right]\mathrm{d}\lambda v(s).$$

Then, we have

$$
\begin{aligned}
&\rho^{\varepsilon}(t)-\rho(t)-\varepsilon\rho_1(t)\\
&=\int_0^t \rho^{\varepsilon}(s)\,h^{\varepsilon}(t)\,\mathrm{d}Y(s)-\varepsilon\int_0^t \rho_1(s)h(s)\,\mathrm{d}Y(s)\\
&\quad-\int_0^t \rho(s)h\left(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\right)\mathrm{d}Y(s)\\
&\quad+\varepsilon\int_0^t \rho(s)\left[A^{\varepsilon}(s)\right]\mathrm{d}Y(s)\\
&=\int_0^t \left(\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right)h^{\varepsilon}(s)\,\mathrm{d}Y(s)\\
&\quad+\int_0^t (\rho(s)+\varepsilon\rho_1(s))\,[h^{\varepsilon}(s)-h(s,x(s)\\
&\quad+\varepsilon\phi(s),u(s)+\varepsilon v(s))]\,\mathrm{d}Y(s)\\
&\quad+\varepsilon\int_0^t \rho_1(s)\,h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))\,\mathrm{d}Y(s)\\
&\quad-\varepsilon\int_0^t \rho_1(s)\,h(s)\,\mathrm{d}Y(s)+\varepsilon\int_0^t \rho(s)\left[A^{\varepsilon}(s)\right]\mathrm{d}Y(s)\\
&=\int_0^t \left(\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right)h^{\varepsilon}(s)\,\mathrm{d}Y(s)\\
&\quad+\int_0^t (\rho(s)+\varepsilon\rho_1(s))\left[B_1^{\varepsilon}(s)\right]\mathrm{d}Y(s)\\
&\quad+\varepsilon\int_0^t \rho_1(s)\left[B_2^{\varepsilon}(s)\right]\mathrm{d}Y(s)\\
&\quad+\varepsilon\int_0^t \rho(s)\left[A^{\varepsilon}(s)\right]\mathrm{d}Y(s),
\end{aligned}
$$

where

$$
\begin{aligned}
B_1^{\varepsilon}(s)&=h^{\varepsilon}(s)-h\left(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s)\right),\\
B_2^{\varepsilon}(s)&=h(s,x(s)+\varepsilon\phi(s),u(s)+\varepsilon v(s))-h(s).
\end{aligned}
$$

Note that

$$
\begin{aligned}
B_1^{\varepsilon}(s)=\int_0^1 &\left[h_x(s,x(s)+\varepsilon\phi(s)+\lambda(x^{\varepsilon}(s)-x(s)\right.\\
&\left.-\varepsilon\phi(s)),v^{\varepsilon}(s))\right]\mathrm{d}\lambda(x^{\varepsilon}(s)-x(s)-\varepsilon\phi(s)).
\end{aligned}
$$

By Lemma 3.3, we know that

$$
\mathbb{E}\int_0^t \left|(\rho(s)+\varepsilon\rho_1(s))B_1^{\varepsilon}(s)\right|^2\,\mathrm{d}s\le C_{\varepsilon}\varepsilon^2, \tag{17}
$$

here C_{ε} denotes some nonnegative constant such that $C_{\varepsilon}\to 0$ as $\varepsilon\to 0$.

Moreover, it is easy to see that

$$
\sup_{0\le t\le T}\mathbb{E}\left[\varepsilon\int_0^t \rho(s)A^{\varepsilon}(s)\,\mathrm{d}Y(s)\right]^2\le C_{\varepsilon}\varepsilon^2, \tag{18}
$$

and

$$
\sup_{0\le t\le T}\mathbb{E}\left[\varepsilon\int_0^t \rho_1(s)B_2^{\varepsilon}(s)\,\mathrm{d}Y(s)\right]^2\le C_{\varepsilon}\varepsilon^2. \tag{19}
$$

From (17), (18) and (19), we get

$$
\begin{aligned}
&\mathbb{E}\left|(\rho^{\varepsilon}(t)-\rho(t))-\varepsilon\rho_1(t)\right|^2\\
&\le C\left[\int_0^t \mathbb{E}\left|(\rho^{\varepsilon}(s)-\rho(s))-\varepsilon\rho_1(s)\right|^2\right.\\
&\quad+\mathbb{E}\int_0^t \left|(\rho(s)+\varepsilon\rho_1(s))B_1^{\varepsilon}(s)\right|^2\,\mathrm{d}s\\
&\quad+\sup_{0\le s\le t}\mathbb{E}\left(\varepsilon\int_0^t \rho(s)A^{\varepsilon}(s)\,\mathrm{d}Y(s)\right)^2\\
&\quad\left.+\sup_{0\le s\le t}\mathbb{E}\left(\varepsilon\int_0^t \rho_1(s)B_2^{\varepsilon}(s)\,\mathrm{d}Y(s)\right)^2\right]\\
&\le C\int_0^t \mathbb{E}\left|\rho^{\varepsilon}(s)-\rho(s)-\varepsilon\rho_1(s)\right|^2\,\mathrm{d}s+C_{\varepsilon}\varepsilon^2.
\end{aligned}
$$

Finally, by using Gronwall's inequality, the proof of Lemma 3.4 is complete. ■

Lemma 3.5: *Let Assumption (H1) hold. Then, we have*

$$
\begin{aligned}
0\le\ &\mathbb{E}\int_0^T \left[\rho_1(t)\,l(t)+\rho(t)\,l_x(t)\phi(t)\right.\\
&\left.+\rho(t)\widehat{\mathbb{E}}\left[\partial_{\mu}l(t)\right]\phi(t)+\rho(t)l_v(t)v(t)\right]\mathrm{d}t\\
&+\mathbb{E}\left[\rho_1(T)\,\psi(x(T),\mathbb{P}_{x(T)})\right]\\
&+\mathbb{E}\left[\rho(T)\,\psi_x(x(T),\mathbb{P}_{x(T)})\phi(T)\right]\\
&+\mathbb{E}\left[\rho(T)\,\widehat{\mathbb{E}}\left[\partial_{\mu}\psi(x(T),\mathbb{P}_{x(T)};\widehat{x}(T))\right]\phi(T)\right].
\end{aligned} \tag{20}
$$

Proof: Using the Taylor expansion, Lemmas 3.3 and 3.4, we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\left[\rho^{\varepsilon}(T)\,\psi(x^{\varepsilon}(T),\mathbb{P}_{x^{\varepsilon}(T)})-\rho(T)\,\psi(x(T),\mathbb{P}_{x(T)})\right]\\
&=\mathbb{E}\left[\rho_1(T)\psi(x(T),\mathbb{P}_{x(T)})+\rho(T)\psi_x(x(T),\mathbb{P}_{x(T)})\phi(T)\right]\\
&\quad+\mathbb{E}\left[\rho(T)\,\widehat{\mathbb{E}}\left[\partial_{\mu}\psi(x(T),\mathbb{P}_{x(T)};\widehat{x}(T))\right]\phi(T)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\int_0^T \left[\rho^{\varepsilon}(t)l^{\varepsilon}(t)-\rho(t)l(t)\right]\mathrm{d}t\\
&=\mathbb{E}\int_0^T \left[\rho_1(t)l(t)+\rho(t)l_x(t)\phi(t)+\rho(t)\widehat{\mathbb{E}}\left[\partial_{\mu}l(t)\right]\widehat{\phi}(t)\right.\\
&\quad\left.+\rho(t)l_v(t)v(t)\right]\mathrm{d}t.
\end{aligned}
$$

Then, by the fact that $\varepsilon^{-1}[J(v^{\varepsilon}(t))-J(u(t))]\ge 0$, we draw the desired conclusion.

Note that

$$
\begin{cases}
\mathrm{d}\widetilde{\rho}(t)=\{h_x(t)\phi(t)+h_v(t)v(t)\}\,\mathrm{d}\widetilde{W}(t),\\
\widetilde{\rho}(0)=0,
\end{cases} \tag{21}
$$

where $\widetilde{\rho}(t)=\rho^{-1}(t)\rho_1(t)$.

By applying Itô's formula to $\Phi(t)\phi(t)$, $y(t)\widetilde{\rho}(t)$ and taking expectation respectively, where $\phi(0)=0$ and $\widetilde{\rho}(0)=0$, we obtain

$$
\begin{aligned}
&\mathbb{E}^u\left[\Phi(T)\,\phi(T)\right]\\
&=\mathbb{E}^u\int_0^T \Phi(t)\,\mathrm{d}\phi(t)+\mathbb{E}^u\int_0^T \phi(t)\,\mathrm{d}\Phi(t)
\end{aligned}
$$

$$+\mathbb{E}^u\int_0^T Q(t)\left[\sigma_x(t)\phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu\sigma(t)\widehat{\phi}(t)\right]+\sigma_v(t)v(t\right]\mathrm{d}t$$
$$+\mathbb{E}^u\int_0^T \overline{Q}(t)\left[c_x(t)\phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu c(t)\widehat{\phi}(t)\right]+c_v(t)v(t)\right]\mathrm{d}t$$
$$+\mathbb{E}^u\int_0^T\int_\Theta R(t,\theta)\left[g_x(t,\theta)\phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta)\widehat{\phi}(t)\right]\right.$$
$$\left.+g_v(t,\theta)v(t)\right]m\ (\mathrm{d}\theta)\,\mathrm{d}t$$
$$=\mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3+\mathbb{I}_4. \tag{22}$$

First, note that

$$\mathbb{I}_1=\mathbb{E}^u\int_0^T\Phi(t)\,\mathrm{d}\phi(t)$$
$$=\mathbb{E}^u\int_0^T\Phi(t)\left[f_x(t)\phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\phi}(t)\right]+f_v(t)v(t)\right]\mathrm{d}t$$
$$=\mathbb{E}^u\int_0^T\Phi(t)f_x(t)\phi(t)\,\mathrm{d}t$$
$$+\mathbb{E}^u\int_0^T\Phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\phi}(t)\right]\mathrm{d}t$$
$$+\mathbb{E}^u\int_0^T\Phi(t)f_v(t)v(t)\,\mathrm{d}t.$$

We proceed to estimate $\mathbb{I}_2$, From Equation (12), we have

$$\mathbb{I}_2=\mathbb{E}^u\int_0^T\phi(t)\,\mathrm{d}\Phi(t)$$
$$=-\mathbb{E}^u\int_0^T\phi(t)\left[f_x(t)\,\Phi(t)\right.$$
$$+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{f}(t)\,\widehat{\Phi}(t)\right]+\sigma_x(t)\,Q(t)$$
$$+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{\sigma}(t)\,\widehat{Q}(t)\right]+c_x(t)\,\overline{Q}(t)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{c}(t)\,\widehat{\overline{Q}}(t)\right]$$
$$+l_x(t)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{l}(t)\right]$$
$$+\int_\Theta\left[g_x(t,\theta)\,R(t,\theta)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{g}(t,\theta)\,\widehat{R}(t,\theta)\right]\right]$$
$$m(\mathrm{d}\theta)+h_x(t)\,K(t)]\,\mathrm{d}t.$$

By simple computation, we have

$$\mathbb{I}_2=-\mathbb{E}^u\int_0^T\phi(t)f_x(t)\,\Phi(t)\,\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{f}(t)\,\widehat{\Phi}(t)\right]\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\phi(t)\,\sigma_x(t)\,Q(t)\,\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{\sigma}(t)\,\widehat{Q}(t)\right]\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\phi(t)\,c_x(t)\,\overline{Q}(t)\,\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{c}(t)\,\widehat{\overline{Q}}(t)\right]\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\phi(t)\,l_x(t)\,\mathrm{d}t-\mathbb{E}^u\int_0^T\phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{l}(t)\right]\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\int_\Theta\phi(t)\,g_x(t,\theta)\,R(t,\theta)\,m(\mathrm{d}\theta)\,\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\int_\Theta\phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{g}(t,\theta)\,\widehat{R}(t,\theta)\right]m(\mathrm{d}\theta)\,\mathrm{d}t$$
$$-\mathbb{E}^u\int_0^T\phi(t)\,h_x(t)\,K(t)\,\mathrm{d}t.$$

Similarly, we can obtain

$$\mathbb{I}_3=\mathbb{E}^u\int_0^T Q(t)\left[\sigma_x(t)\phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu\sigma(t)\widehat{\phi}(t)\right]+\sigma_v(t)v(t)\right]\mathrm{d}t$$
$$+\mathbb{E}^u\int_0^T\overline{Q}(t)\left[c_x(t)\phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu c(t)\widehat{\phi}(t)\right]+c_v(t)v(t)\right]\mathrm{d}t,$$

and

$$\mathbb{I}_4=\mathbb{E}^u\int_0^T\int_\Theta R(t,\theta)\left[g_x(t,\theta)\phi(t)+\widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta)\,\widehat{\phi}(t)\right]\right.$$
$$\left.+g_v(t,\theta)v(t)\right]m(\mathrm{d}\theta)\,\mathrm{d}t.$$

Then, applying Itô's formula to $y(t)\widetilde{\rho}(t)$ and taking expectation, we get

$$\mathbb{E}^u\left[y(T)\,\widetilde{\rho}(T)\right]=\mathbb{E}^u\int_0^T y(t)\,\mathrm{d}\widetilde{\rho}(t)+\mathbb{E}^u\int_0^T\widetilde{\rho}(t)\,\mathrm{d}y(t)$$
$$+\mathbb{E}^u\int_0^T K(t)\left\{h_x(t)\phi(t)+h_v(t)v(t)\right\}\mathrm{d}t$$
$$=\mathbb{J}_1+\mathbb{J}_2+\mathbb{J}_3, \tag{23}$$

where $\mathbb{J}_1=\mathbb{E}^u\int_0^T y(t)\,\mathrm{d}\widetilde{\rho}(t)$ is a martingale with zero expectation. Moreover, by a simple computations, we get

$$\mathbb{J}_2=\mathbb{E}^u\int_0^T\widetilde{\rho}(t)\,\mathrm{d}y(t)=-\mathbb{E}^u\int_0^T\widetilde{\rho}(t)\,l(t)\,\mathrm{d}t,$$

and

$$\mathbb{J}_3=\mathbb{E}^u\int_0^T K(t)\left[h_x(t)\phi(t)+h_v(t)v(t)\right]\mathrm{d}t.$$

Now, by applying Fubini's theorem, we obtain

$$\mathbb{E}^u\int_0^T\Phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{f}(t)\,\widehat{\phi}(t)\right]\mathrm{d}t$$
$$=\mathbb{E}^u\int_0^T\phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\Phi}(t)\right]\mathrm{d}t, \tag{24}$$

$$\mathbb{E}^u\int_0^T Q(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{\sigma}(t)\,\widehat{\phi}(t)\right]\mathrm{d}t$$
$$=\mathbb{E}^u\int_0^T\phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\sigma(t)\widehat{Q}(t)\right]\mathrm{d}t, \tag{25}$$

$$\mathbb{E}^u\int_0^T\overline{Q}(t)\,\widehat{\mathbb{E}}\left[\partial_\mu\widehat{c}(t)\,\widehat{\phi}(t)\right]\mathrm{d}t$$

$$= \mathbb{E}^u \int_0^T \phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu c(t)\widehat{\widehat{Q}}(t)\right] \mathrm{d}t, \tag{26}$$

and

$$\mathbb{E}^u \int_0^T \int_\Theta R(t,\theta)\,\widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(t,\theta)\,\widehat{\phi}(t)\right] m(\mathrm{d}\theta)\,\mathrm{d}t$$
$$= \mathbb{E}^u \int_0^T \int_\Theta \phi(t)\,\widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta)\widehat{R}(t,\theta)\right] m(\mathrm{d}\theta)\,\mathrm{d}t. \tag{27}$$

Finally, substituting (22), (23), (24), (25), (26) and (27) into (20), this completes the proof of Theorem 3.1. ■

4. Partially observed McKean–Vlasov linear quadratic control problem with jumps

In this section, as an application, we study partially observed optimal control problem for Mckean–Vlasov linear quadratic control problem with jump diffusion, where the stochastic system is described by a set of linear McKean–Vlasov stochastic differential equations and the cost is described by a quadratic function.

By applying our stochastic maximum principle established in Section 3 and classical filtering theory, we obtain an explicit expression of the optimal control represented in feedback form involving both controlled state process $x(t)$ as well as its law represented by $\mathbb{E}[x(t)]$ via the solutions of ordinary differential equations (ODEs).

Consider the following partially observed control system:

$$\begin{cases} \mathrm{d}x^v(t) = f\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right)\mathrm{d}t \\ \qquad + \sigma\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right)\mathrm{d}W(t) \\ \qquad + c\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right)\mathrm{d}\widetilde{W}(t) \\ \qquad + \int_\Theta g\left(t, x^v(t_-), \mathbb{P}_{x^v(t_-)}, v(t), \theta\right)\widetilde{N}(\mathrm{d}\theta, \mathrm{d}t) \\ x^v(0) = x_0, \end{cases} \tag{28}$$

where

$$f\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) = A(t)\,x(t) + B(t)\,\mathbb{E}[x(t)] + C(t)\,v(t),$$
$$\sigma\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) = D(t),$$
$$c\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right) = 0,$$
$$g\left(t, x^v(t_-), \mathbb{P}_{x^v(t-)}, v(t), \theta\right) = F(t),$$
$$h(t, x^v(t),\ v(t)) = G(t),$$

with an observation

$$\begin{cases} \mathrm{d}Y(t) = G(t)\,\mathrm{d}t + \mathrm{d}\widetilde{W}(t), \\ Y(0) = 0, \end{cases} \tag{29}$$

and the quadratic cost functional

$$J(v(\cdot)) = \mathbb{E}^u\left[\int_0^T L(t)\,v^2(t)\,\mathrm{d}t + M_T x^2(T)\right]. \tag{30}$$

Here, the coefficients $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $F(\cdot)$, $G(\cdot)$ and $L(\cdot)$ are bounded continuous functions and $M_T \geq 0$. For any $v \in \mathcal{U}_{ad}([0,T])$, Equations (28) and (29) have a unique solutions respectively.

Our goal is to find an explicitly optimal control to minimise the cost functional $J(v(\cdot))$ over $v(\cdot) \in \mathcal{U}_{ad}([0,T])$, subject to (28) and (29).

Now, we begin to seek the explicit expression of the optimal control by two steps.

First step. Find optimal control.

We start by write down the Hamiltonian function H:

$$\begin{aligned} H(t,x,v,\Phi,Q,\overline{Q},R(\cdot)) = {} & [A(t)\,x(t) + B(t)\,\mathbb{E}[x(t)] \\ & + C(t)\,v(t)]\,\Phi(t) + D(t)\,Q(t) \\ & + G(t)\,K(t) + L(t)\,v^2(t) \\ & + \int_\Theta F(t)\,R(t,\theta)\,m(\mathrm{d}\theta), \end{aligned} \tag{31}$$

where $x(\cdot)$ is the optimal trajectory, solution of Equation (28) corresponding to the optimal control $u(\cdot)$.

By Theorem 3.1 and (31), the optimal control $u(\cdot)$ satisfies the following expression:

$$u(t) = -\frac{1}{2}L^{-1}(t)\,C(t)\,\mathbb{E}\left[\Phi(t) \mid \mathcal{F}_t^Y\right], \tag{32}$$

where $(\Phi(\cdot), Q(\cdot), \overline{Q}(\cdot), R(\cdot,\cdot))$ is the solution of the following BSDE:

$$\begin{cases} -\mathrm{d}\Phi(t) = [A(t)\,\Phi(t) + B(t)\,\mathbb{E}[\Phi(t)]]\,\mathrm{d}t \\ \qquad - Q(t)\,\mathrm{d}W(t) - \overline{Q}(t)\,\mathrm{d}\widetilde{W}(t) \\ \qquad - \int_\Theta R(t,\theta)\,\mathrm{d}\widetilde{N}(\mathrm{d}\theta, \mathrm{d}t), \\ \Phi(T) = 2M_T x(T). \end{cases} \tag{33}$$

Second step. Give the explicit expression of the optimal control in (32).

From Liptser and Shiryayev (1977) and Xiong (2008), we can deduce the following group of filtering equations:

$$\begin{cases} \mathrm{d}\widehat{x}(t) = \left[A(t)\,\widehat{x}(t) + B(t)\,\mathbb{E}[\widehat{x}(t)] - \frac{1}{2}L^{-1}(t)\,C^2(t)\,\widehat{\Phi}(t)\right]\mathrm{d}t \\ -\mathrm{d}\widehat{\Phi}(t) = \left[A(t)\,\widehat{\Phi}(t) + B(t)\,\mathbb{E}\left[\widehat{\Phi}(t)\right]\right]\mathrm{d}t - \widehat{\overline{Q}}(t)\,\mathrm{d}\widetilde{W}(t), \\ \widehat{x}(0) = x_0,\ \widehat{\Phi}(T) = 2M_T\widehat{x}(T),\ \widehat{\overline{Q}}(t) = 0, \end{cases} \tag{34}$$

where $\widehat{\xi}(t) = \mathbb{E}^u[\xi(t) \mid \mathcal{F}_t^Y]$ is the filtering estimate of the state $\xi(t)$ depending on the observable filtration $\mathcal{F}_t^Y$, $\xi = x, \Phi, \overline{Q}$.

Now, to solve the above Equation (34), we conjecture a process $\widehat{\Phi}(\cdot)$ of the form

$$\widehat{\Phi}(t) = \varphi(t)\,\widehat{x}(t) + \psi(t)\,\mathbb{E}[\widehat{x}(t)], \tag{35}$$

where $\varphi(\cdot)$, $\psi(\cdot)$ are deterministic differential functions.

We derive (35) and comparing it with (34), we get

$$\begin{aligned} & -\{A(t)\,(\varphi(t)\,\widehat{x}(t) + \psi(t)\,\mathbb{E}[\widehat{x}(t)]) \\ & \quad + B(t)\,\mathbb{E}[\varphi(t)\,\widehat{x}(t) + \psi(t)\,\mathbb{E}[\widehat{x}(t)]]\} \\ & \quad = \dot{\varphi}(t)\,\widehat{x}(t) + \dot{\psi}(t)\,\mathbb{E}[\widehat{x}(t)] \end{aligned}$$

$$+\varphi(t)\left\{A(t)\widehat{x}(t)+B(t)\mathbb{E}[\widehat{x}(t)]\right.$$
$$\left.-\frac{1}{2}L^{-1}(t)C^2(t)(\varphi(t)\widehat{x}(t)+\psi(t)\mathbb{E}[\widehat{x}(t)])\right\}$$
$$+\psi(t)\left\{(A(t)+B(t))\mathbb{E}[\widehat{x}(t)]-\frac{1}{2}L^{-1}(t)C^2(t)\right.$$
$$\left.\times\mathbb{E}[\varphi(t)\widehat{x}(t)+\psi(t)\mathbb{E}[\widehat{x}(t)]]\right\}. \tag{36}$$

By comparing the coefficients of $\widehat{x}(t)$ and $\mathbb{E}[\widehat{x}(t)]$ in (36), we get the following ODEs:

$$\begin{cases}\dot{\varphi}(t)+2A(t)\varphi(t)-\frac{1}{2}L^{-1}(t)C^2(t)\varphi^2(t)=0,\\ \varphi(T)=2M_T,\end{cases} \tag{37}$$

and

$$\begin{cases}\dot{\psi}(t)+2(A(t)+B(t))\psi(t)+2B(t)\varphi(t)\\ -L^{-1}(t)C^2(t)\varphi(t)\psi(t)-\frac{1}{2}L^{-1}(t)C^2(t)\psi^2(t)=0,\\ \psi(T)=0.\end{cases} \tag{38}$$

Note that Equations (37) and (38) are Bernoulli differential equation and Riccati differential equation respectively. To solve (37) and (38), we can use the similar method in Lakhdari et al. (2021). Then, the optimal control $u(\cdot)\in\mathcal{U}_{ad}([0,T])$ for the problem (30) is given in the feedback form

$$u(t,\widehat{x}(t))=-\frac{1}{2}L^{-1}(t)C(t)[\varphi(t)\widehat{x}(t)+\psi(t)\mathbb{E}[\widehat{x}(t)]],$$

where $\varphi(\cdot),\psi(\cdot)$ determined by (37) and (38) respectively.

5. Conclusion

In this paper, we have developed the necessary conditions for partially observed stochastic optimal control problem, where the controlled state process is governed by general McKean–Vlasov differential equations with jumps. By transforming the partial observation problem to a related problem with full information, a stochastic maximum principle for optimal control has been established via the derivative with respect to probability measure. A partially observed linear-quadratic control problem with jumps has been solved explicitly to illustrate our theoretical results. The main feature of these results is to explicitly solve some mathematical finance problems such as conditional mean–variance portfolio selection problem in incomplete market. Apparently, there are many problems left unsolved, and one possible problem is to establish some optimality conditions for partially observed stochastic optimal control for systems described by forward–backward stochastic differential equations of general McKean–Vlasov type with jumps with some applications.

Acknowledgements

The authors would like to thank the editor and anonymous referees for their constructive corrections and valuable suggestions that improved the manuscript considerably.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Shahlar Meherrem https://orcid.org/0000-0002-4255-4760
Mokhtar Hafayed https://orcid.org/0000-0002-8915-9530

References

Bouchard, B., & Elie, R. (2008). Discrete time approximation of decoupled forward-backward SDE with jumps. *Stochastic Processes and Their Applications*, *118*(1), 53–75. https://doi.org/10.1016/j.spa.2007.03.010

Buckdahn, R., Li, J., & Ma, J. (2016). A stochastic maximum principle for general mean-field systems. *Applied Mathematics and Optimisation*, *74*(3), 507–534. https://doi.org/10.1007/s00245-016-9394-9

Djehiche, B., & Tembine, H. (2016). Risk sensitive mean-field type control under partial observation. In *Stochastics of environmental and financial economics* (pp. 243–263). Springer.

Fleming, W. H. (1968). Optimal control of partially observable diffusions. *SIAM Journal on Control*, *6*(2), 194–214. https://doi.org/10.1137/0306015

Guenane, L., Hafayed, M., Meherrem, S., & Abbas, S. (2020). On optimal solutions of general continuous-singular stochastic control problem of McKean–Vlasov type. *Mathematical Methods in the Applied Sciences*, *43*(10), 6498–6516. https://doi.org/10.1002/mma.v43.10

Hafayed, M., Meherrem, S., Eren, Ş., & Guçoglu, D. H. (2018). On optimal singular control problem for general Mckean–Vlasov differential equations: necessary and sufficient optimality conditions. *Optimal Control Applications & Methods*, *39*(3), 1202–1219. https://doi.org/10.1002/oca.v39.3

Kac, M. (1959). Foundations of kinetic theory. University of California Press (pp. 171–197).

Lakhdari, I. E., Miloudi, H., & Hafayed, M. (2021). Stochastic maximum principle for partially observed optimal control problems of general McKean–Vlasov differential equations. *Bulletin of the Iranian Mathematical Society*, *47*(4), 1021–1043. https://doi.org/10.1007/s41980-020-00426-1

Lions, P. L. (2013). Cours au Collège de France: Théorie des jeu à champs moyens. http://www.college-de-france.fr/default/EN/all/equ[1]der/audiovideo.jsp.

Liptser, R. S., & Shiryayev, A. N. (1977). *Statistics of random process*. Springer.

McKean, J. R. H. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proceedings of the National Academy of Sciences of the United States of America*, *56*(6), 1907–1911. https://doi.org/10.1073/pnas.56.6.1907

Meherrem, S., & Hafayed, M. (2019). Maximum principle for optimal control of McKean–Vlasov FBSDEs with Lévy process via the differentiability with respect to probability law. *Optimal Control Applications & Methods*, *40*(3), 499–516. https://doi.org/10.1002/oca.v40.3

Tang, M., & Meng, Q. X. (2017). Maximum principle for partial observed zero-sum stochastic differential game of mean-field SDEs. In *36th Chinese control conference (CCC)* (pp. 1868–1875). https://doi.org/10.23919/ChiCC.2017.8027625.

Wang, G., & Wu, Z. (2009). General maximum principles for partially observed risk-sensitive optimal control problems and applications to finance. *Journal of Optimization Theory and Applications*, *141*(3), 677–700. https://doi.org/10.1007/s10957-008-9484-1

Wang, G., Wu, Z., & Xiong, J. (2013). Maximum principles for forward–backward stochastic control systems with correlated state and observation noise. *SIAM Journal on Control and Optimization*, *51*(1), 491–524. https://doi.org/10.1137/110846920

Wang, G., Wu, Z., & Xiong, J. (2015). A linear-quadratic optimal control problem of forward–backward stochastic differential equations with partial information. *IEEE Transactions on Automatic Control*, *60*(11), 2904–2916. https://doi.org/10.1109/TAC.2015.2411871

Wang, G., Wu, Z., & Zhang, C. (2014, July 28–30). Maximum principles for partially observed mean-field stochastic systems with application to financial engineering. In *Proceedings of the 33rd Chinese control conference* (pp. 5357–5362). IEEEConference. https://doi.org/10.1109/ChiCC.2014.6895853.

Wang, M., Shi, Q., & Meng, Q. (2019). Optimal control of forward–backward stochastic jump-diffusion differential systems with observation noises: stochastic maximum principle. *Asian Journal of Control*, *23*(1), 241–254. https://doi.org/10.1002/asjc.v23.1

Wang, G., Zhang, C., & Zhang, W. (2014). Stochastic maximum principle for mean-field type optimal control with partial information. *IEEE Transactions on Automatic Control*, *59*(2), 522–528. https://doi.org/10.1109/TAC.2013.2273265

Wu, Z., & Zhuang, Y. (2018). Partially observed time-inconsistent stochastic linear-quadratic control with random jumps. *Optimal Control Applications & Methods*, *39*(1), 230–247. https://doi.org/10.1002/oca.v39.1

Xiao, H. (2013). Optimality conditions for optimal control of jump-diffusion SDEs with correlated observations noises. *Mathematical Problems in Engineering*, *2013*, 613159. https://doi.org/10.1155/2013/613159

Xiong, J. (2008). *An introduction to stochastic filtering theory* (Vol. 18). Oxford University Press on Demand.

Zhang, S., Xiong, J., & Liu, X. (2018). Stochastic maximum principle for partially observed forward–backward stochastic differential equations with jumps and regime switching. *Science China Information Sciences*, *61*(7), 70211. https://doi.org/10.1007/s11432-017-9267-0

Appendix

Proposition A.1: *Let $\mathcal{G}$ be the predictable σ-field on $\Omega \times [0,T]$, and f be a $\mathcal{G} \times \mathcal{B}(\Theta)$-measurable function such that*

$$\mathbb{E}\int_0^T \int_\Theta \left|f(r,\theta)\right|^2 m(\mathrm{d}\theta)\,\mathrm{d}r < \infty,$$

then for all $p \geq 2$ there exists a positive constant $C = C(T,p,m(\Theta))$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \int_\Theta f(r,\theta)\,N(\mathrm{d}\theta,\mathrm{d}r)\right|^p\right] < C\mathbb{E}\left[\int_0^T \int_\Theta \left|f(r,\theta)\right|^p m(\mathrm{d}\theta)\,\mathrm{d}r\right].$$

Proof: See Bouchard and Elie (2008, Appendix). ■