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OPTIMAL CONTROL FOR HYBRID SYSTEM

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ABSTRACT

OPTIMAL CONTROL FOR HYBRID SYSTEM

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This thesis includes a different approach for solving optimal control for switched systems. We focus on problems in which a prespecified sequence of active subsystem is given. For these problems we must have optimal switching instants and optimal continuous inputs. For this reason the derivatives of optimal cost with respect the switching instants need to be known.

Also in thesis an approach for solving optimal control problems of switched system.In general,in such problems one needs to find optimal continuous inputs and optimal switching sequences. After formulating a general optimal control problem, we study two stage methodology. Since many practical problems only concern optimization where the number of switchings and the sequence of active subsystems are given, we think about on such problems and propose a method which uses nonlinear optimization and is based on direct differantiations of value functions.

It is also developed a computational method for solving an optimal control problem which is governed by a switched dynamical system with time delay. Our approach is to parametrize the switching instants as a new parameter vector to be optimized. Then,we derive the required the gradient of the cost function which is obtained via solving a number of delay differential equations forward in time. Finally, there are given optimality condition for the switching control system.

Keywords: Switched system, Delay, Parametrization, Optimal control.

YEMİN METNİ

Yüksek Lisans tezi olarak sunduğum ''Hybrid System for Optimal Control'' adlı çalışmanın, tarafımdan bilimsel ahlak ve geleneklere aykırı düşecek bir yardıma başvurmaksızın yazıldığının ve yararlandığım eserlerin ''Bibliography'' bölümünde gösterilenlerden oluştuğunu, bunlara atıf yapılarak yararlanılmış olduğunu belirtir ve bunu onurumla doğrularım.

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TEŞEKKÜR

Yüksek lisans tezimi hazırlarken bana rehberlik eden ve desteğini eksik etmeyen danışman hocam Yrd. Doç. Dr. Shahlar MEHERREM'e , lisans ve yüksek lisans eğitimimde ders aldığım tüm hocalarıma, hazırlık aşamasında yardımcı olan tüm sevdiklerime teşekkür ederim.

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İÇİNDEKİLER

INTRODUCTION

Switched system are class of hybrid systems which consists of several subsystems and switching law orchestrating the active subsystemat each time instant.

Optimal control problems for switched systems ,which require the solutions of both the optimal switching sequences and the optimal continuous input. Many results,which report progress regarding the theoretical or practical issues for continuos time or discrete time versions of such problems, appeared the literature [1],[5],[6]

Maximum principle and Hamiltonian Jacobi-Bellman equation for hybrid and switched systems derived in literature [4].Optimal control problems of hybrid and switched systems have been attracting researches from various in science and engineering significance in theory and application,classifiedto categories, theoretical and practical. These results extended classical maximum principle or dynamic programming approach for problems. Also, proves a maximum principle for hybrid system with autonoumus switching only. Another result is proof of existence of optimal control for system with two subsystems. Complicated versions of maximum principle under additional assumptions.

The problem formulations and methodologiesare very diverse in this category It is important that in thesis have different models and optimal control objarctives for hybrid system. This thesis presents solving optimal control problems of switched systems.

This thesis presents solving of optimal control problems for switched systems. In thesis focused on problem which a presrecified sequence of active subsystems is given. From here, in thesis, need to seek optimal instants ans optimal continuous inputs. In order to redsearch for optimal switching instants,the derivatives of the optimal cost need to be known. It is important that, method is trancribes an optimal control problem into an equivalent parametrized by the switching instants and derives the derivatives based on solution of a two boundary value formed by state, costate, stationary equations, the boundary and continuity conditions with their differentiations.

In the thesis our approach is to parametrize the switching instants a new parameter vector to be optimized. After,derived gradient of cost function which is obtained via solving a number of delay differential equations in time.

NECESSARY OPTIMALITY CONDITIONS FOR SWITCHED SYSTEM

1.1 Switched Systems

We think about switched systems consisting of the subsystems

$\dot{x} = f_i\ (x, u) \qquad f_i: R^n \times R^m \rightarrow R^n \qquad i \in I \triangleq \{1,2, \dots, M\}$ (1.1.1)

From control switched system we must have continuous input and switching sequence.

A switching sequence in t $\in [t_0, t_f]$ regulates the sequence of active subsystems and is defined as $((t_0, i_0), \dots, (t_k, i_k))$
where $0 < K < \infty, t_0 \leq \cdots \leq t_k$ and $i_k \in I$ (1.1.2)

for k=$0,1, \dots, K$.

Note that, (t_k, i_k) indicates that at instant t_k the system switches from subsystem i_k to $i_{k-1,}$ during the time interval $[t_k, t_{k+1}]$ $([t_{k,} t_f\ if\ k = K)$ subsystem i_k is active. For switched systems we consider nonZeno sequences which switch at most finite number of times in $[t_0, t_f]$. Also we regard σ which discrete input, then we have control input u with (σ, u).

Finally, switching system from general hybrid system is continuous state and does not jumps at switching instants.

Optimal control problem

In the sequel, $U_{[t_0, t_f]} \triangleq$, which means that $\{u | u \in c_p[t_0, t_f], u(t) \in R^m\}$. We concentrate on problems which involve optimizations and prespecified sequence of active subsystems.

Problem 1: Think about switched systems which consists of subsystems $\dot{x} = f_i(x, u)\ i \in I$. Given a fixed time interval $[t_0, t_f]$ and a prespecified sequence of active subsystems $(i'_0, \dots, i_k)$, find a continuous input $u \in U_{[t_0, t_f]}$ and a switching instants $t_1, \dots, t_k$ such that the corresponding continuous state trajectory x departs from a given initial state

$x(t_0) = x_0\ , S_f = \{x | \phi_f(x) = 0, \phi_f: R^n \rightarrow R^{l_f}\}$ and the cost functional

$J = \psi\left(x(t_f)\right) + \int_{t_0}^{t_f} L(x(t), u(t) dt)$ (1.1.3)
is minimized.

In order to solve problem 1, needs to also nonlinear optimization techniques.

Problem 1 is basic optimal control of Bolza form. In the sequel, we assume that f, L are continuous and have continuous partial derivatives with respect the x; ϕ_f is continuosly differentiable; ψ has twice continuous derivatives. We formulate Problem

1 with a fixed final time is mainly fort he convenience of subsequent studies. For with free final time t_f, we can introduce an additional state variable and transcribe to fixed final time problem.

Two stage decomposition

Stage (a) is conventional optimal control problem which is research the minimum value of J with respect the u under given switching sequence $\sigma = \big((t_0, i_0), (t_1, i_1) \dots, (t_k, i_k)\big)$. In the sequel,we define the coresponding optimal cost function $J_1(\hat{t})$, $\hat{t} \triangleq \{t_1, \dots, t_K\}^T$.

Stage (b) is constrained nonlinear optimization problem $min_t\, J_1(\hat{t})$, $J_1(\hat{t}) \in T$

(1.1.4)

$\{\hat{t} = (t_1, .., t_K)^T | t_0 \le t_1 \le \dots \le t_K \le t_f\}$. In order to solve Problem 1, we need nonlinear optimization techniques.

Stage(a): We need to find an optimal continuos input u and minimum J. Although different subsystems are active in different time intervals, stage(a) research $J_1(\hat{t})$ for $\hat{t} = (t_1, .., t_K)^T$ is conventional intevals are fixed.

Theorem 1: Necessary conditions for stage (a): Think about the stage (a) problem for problem

1. Assume that subsystem k is active in $[t_{k-1}, t_k)$ for $1 \le k \le K$ and subsystem $K+1$ in $[t_K, t_{K+1}]$ with $t_{K+1} = t_f$. Let $u \epsilon U_{[t_0, t_f]}$ be a continuous input such that the coresponding continuous state trajectory x departs from given $x(t_0) = x_0$ and $S_f = \{x | \phi_f(x) = 0, \phi_f: R^n \to R^{l_f}\}$. In order for u to be optimal it is necessary that vector function $p(t) = [p_1(t), \dots, p_n(t)], t \in [t_0, t_f]$ such that following conditions hold;

a) For almost any $t \in [t_0, t_f]$ the following state and costate equations hold:

state equation: $\left(\frac{\partial H}{\partial p}(x(t), p(t), u(t)\right)^T$ (1.1.5)

costate equation: $\frac{dp(t)}{dt} = -\left(\frac{\partial H}{\partial x}\big(x(t), p(t), u(t)\big)\right)^T$ (1.1.6)

Here $H(x, p, u) \triangleq L(x, u) + p^T f_k(x, u), t \in [t_{K-1}, t_k)$ if $k = K$ if $t \in [t_K, t_F]$.

b) For almost any $t \in [t_0, t_f]$ the stationary condition holds:

$$\left(\frac{\partial H}{\partial x}\left(x(t), p(t), u(t)\right)\right)^T \tag{1.1.7}$$

c) At t_f $p(t_f) = \left(\frac{\partial \psi}{\partial x}\left(x(t_f)\right)\right)^T + \left(\frac{\partial \phi_f}{\partial x}\left(x(t_f)\right)\right)^T \lambda$ (1.1.8)

where λ is an l_f dimensional vector.

d) At any t_k, $k = 1,2 \ldots, K$, we have $p(t_k -) = p(t_k +)$ (1.1.9)

Proof: Using Lagrange multipliers to adjoin the constraints

$\dot{x} = f_k(x,u), k = 1, \ldots, K + 1$ and $\phi_f\left(x(t_f)\right) = 0$ to J. The augmented performance index is

$J' = \psi\left(x(t_f)\right) + \lambda^T \phi_f\left(x(t_f)\right) + \sum_{k=1}^{K+1} \int \left((L(x,u) + p^T(t)(f_k(x,u) - \dot{x})\right)dt$ by

$H(x,p,u) \triangleq L(x,u) + p^T f_k(x,u), t \in [t_{K-1}, t_k), 1 \leq k \ldots . \leq K$ and $t \in [t_K, t_{K+1}]$ with

$t_{K+1} = t_f$ if $k = K+1 \rightarrow J' = \psi\left(x(t_f)\right) + \lambda^T \phi_f\left(x(t_f)\right) + \sum_{k=1}^{K+1}\left((H(u,u,p) - p^T \dot{x}\right)dt$, from calculus of variations

$\delta J' = \left(\frac{\partial \psi}{\partial x}\left(x(t_f)\right) + \lambda^T \frac{\partial \phi_f}{\partial x}\left(x(t_f)\right) - p^T(t_f)\right)\delta x(t_f) + \sum_{k=1}^{K} p^T(\, t_k+) - p^T(t_k-)\delta x(t_k) \sum_{k=1}^{K} \int_{t_{k=1}}^{t_k} \left(\left(\frac{\partial H}{\partial x} + \dot{p}^T\right)\delta x\right) + \frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial p} - \dot{x}^T\right)\delta p dt.$

From Lagrange theory a necessary condition for a solution to be optimal is $\delta J'$=0. Setting the zero the coefficients of the independent increments $\delta x(t_f), \delta x(t_k)'s$, $\delta x, \delta u, \delta p$ yields the necessary conditions a)-d). □

The conditions a)-d) present a two boundary value differential algebraic equation (DAE), which solved numerical methods.

Stage(b): We need to solve the constrained nonlinear optimization problem(4) with simple constraints. Computational methods for finding local optimal solutions of such problems are abundant in nonlinear optimization literature.

1.2 Approach Based on Parametrization of Switching Instants

In thesis an approach to problem 1 based on parametrization of the switching instants is presented. The first step is transcribe an optimal control problem into an equivalent conventional optimal control problem parametrized by the switching instants.

Equivalent problem formulation

Here, defined the transcription of problem 1 into an qeuivalent problem parametrized by the unknown switching instants also switching instants are fixed with respect to new independent time variable. We think about two subsystems where subsystem 1 is active in $t \epsilon [t_0, t_1)$ and subsystem 2 is active in $t\epsilon[t_1, t_f]$ and also $S_f = R^n$.

Problem 2: For switched system $\dot{x} = f_1(x,u)$ $\quad t_0 \le t \le t_1$ (1.2.1)

$\dot{x} = f_2(x,u)$ $\quad t_1 \le t \le t_f$ (1.2.2)

Find a switching instant t_1 and a continuos input $u(t), t \in [t_0, t_f]$ suh that

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x,u)dt \quad (1.2.3)$$

is minimized.

For transcribe equivalent problem 2 we define a state variable x_{n+1} coresponding switching instant t_1.

Let x_{n+1} satisfy $\frac{dx_{n+1}}{dt} = 0$ (1.2.4)

$x_{n+1}(0) = t_1$ (1.2.5)

Next, we introduce τ. A piecewise linear relation ship between t and τ is $t = t_0 + (x_{n+1} - t_0)\lambda,\ 0 \le \tau \le 1$ and $t = \{(x_{n+1} + (t_f - x_{n+1})(\tau - 1)\}$ for $1 \le \tau \le 2$ (1.2.6)

By introducing x_{n+1} and, τ problem 2 is transcribe into following problem.

Problem 3 (Equivalent problem) : For system with dynamics

$$\frac{dx(\tau)}{d\tau} = (x_{n+1} - t_0)f_1(x,u) \quad (1.2.7)$$

and

$$\frac{dx_{n+1}}{d\tau} = 0 \quad \text{for} \quad \tau\epsilon[0,1) \quad (1.2.8)$$

and

$$\frac{dx(\tau)}{d\tau} = (x_{n+1} - t_0)f_1(x,u) \quad (1.2.9)$$

$$\frac{dx_{n+1}}{d\tau} = 0 \quad (1.2.10)$$

in $\tau \in [1,2]$, find x_{n+1} and $u(\tau), \tau\epsilon[0,2]$ such that;

$$J = \psi(x(2)) + \int_0^1 (x_{n+1} - t_0)L(x,u)d\tau + \int_1^2 \int_0^1 (x_{n+1} - t_0)L(x,u)d\tau \quad (1.2.11)$$

Note that problem 2 and problem 3 are equivalemt in optimal solution for problem 3 is an optimal solution for problem 2 by proper change of independent variable and by regarding $x_{n+1} = t_1$.

Method based on solving a boundary value differential algebraic equation

We develop a method for deriving accurate numerical value $\frac{dJ_1}{dt_1}$. The method is based on the solution of a two boundary value DAE formed by state, costate, stationary equations, the boundary and continuity conditions for problem 3, along with their derivatives with respect to parameter x_{n+1}.

Think about the equivalent Problem 3, define

$$\tilde{f}_1(x,u,\ x_{n+1}) \triangleq (x_{n+1}-t_0)f_1(x,u) \tag{1.2.12}$$

$$\tilde{f}_2(x,u,x_{n+1}) \triangleq (x_{n+1}-t_0)f_2(x,u) \tag{1.2.13}$$

$$\tilde{L}_1(x,u,\ x_{n+1}) \triangleq (x_{n+1}-t_0)L(x,u) \tag{1.2.14}$$

$$\tilde{L}_2(x,u,\ x_{n+1}) \triangleq (x_{n+1}-t_0)L(x,u) \tag{1.2.15}$$

Consequently we denote it as$x(\tau,\ x_{n+1})$. We define the parametrized Hamiltonian as

$$H(x,p,u,x_{n+1}) \triangleq \{\tilde{L}_1(x,u,x_{n+1})+p^T\ \tilde{f}_1(x,u,x_{n+1}) \quad \tau\in[0,1)$$
$$H(x,p,u,x_{n+1}) \triangleq \{\tilde{L}_2(x,u,x_{n+1})+p^T\ \tilde{f}_2(x,u,x_{n+1}) \quad \tau\in[1,2]$$
(1.2.16)

The necessary conditions a) and b) provide us with following state, costate, stationary equations:

$$\frac{\partial x}{\partial \tau}=\left(\frac{\partial H}{\partial p}\right)^T=\tilde{f}_k(x,u,x_{n+1}) \tag{1.2.17}$$

$$\frac{\partial p}{\partial \tau}=-\left(\frac{\partial H}{\partial x}\right)^T=-\left(\frac{\partial \tilde{f}_k}{\partial x}\right)^T p-\left(\frac{\partial \tilde{L}_k}{\partial x}\right)^T \tag{1.2.18}$$

$$0=\left(\frac{\partial H}{\partial u}\right)^T=\left(\frac{\partial \tilde{f}_k}{\partial x}\right)^T p+\left(\frac{\partial \tilde{L}_k}{\partial x}\right)^T \tag{1.2.19}$$

Note that p and u are coresponding to optimal solution are also functions of τ and x_{n+1}.

From the necessary condition c) of theorem 1, we have

$$x(0,x_{n+1})=x_0 \tag{1.2.20}$$

$$p(2,x_{n+1})=\left(\frac{\partial \psi}{\partial x}x((2,x_{n+1}))\right)^T \tag{1.2.21}$$

the necessary condition d) tell us $p(\tau,x_{n+1})$ is continuous at $\tau=1$ for fixed x_{n+1}. (1.2.22)

Then we have optimal value of J which is function of the parameter x_{n+1},

$$J_1(x_{n+1}) = \psi\big(x(2, x_{n+1})\big) + \int_0^1 \tilde{L}_1(x, u, x_{n+1}) d\tau + \int_0^1 \tilde{L}_2(x, u, x_{n+1}) d\tau$$

(1.2.23)

$$\frac{dJ_1}{dx_{n+1}} = \frac{\partial\psi\big(x(2,x_{n+1})\big)}{\partial x}\frac{\partial x(2,x_{n+1})}{\partial x_{n+1}} + \int_0^1\left((L(x,u) + x_{n+1} - t_0)\left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial x_{n+1}}\right) + \left(\frac{\partial L}{\partial u}\ \frac{\partial \mathrm{u}}{\partial \mathrm{x}_{\mathrm{n+1}}}\right)\right) d\tau + \int_1^2\left(-L(x,u) + t_f - x_{n+1} \times \left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial x_{n+1}}\right) + \left(\frac{\partial L}{\partial u}\ \frac{\partial \mathrm{u}}{\partial \mathrm{x}_{\mathrm{n+1}}}\right)\right) d\tau$$

(1.2.24)

.

So must have $\partial x(\tau, x_{n+1})$ and $\frac{\partial(\tau, x_{n+1})}{\partial x_{n+1}}$ (x_{n+1} is fixed) in order to the value $\frac{dJ_1}{dx_{n+1}}$.

Then; $\frac{\partial}{\partial\tau}\left(\frac{\partial x}{\partial x_{n+1}}\right) = \frac{\partial}{\partial x_{n+1}}\left(\frac{x\partial}{\partial\tau}\right) = f_1 + (x_{n+1} - t_0) \times \left(\frac{\partial f_1}{\partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x_{n+1}}\right)$

(1.2.25)

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p}{\partial x_{n+1}}\right) = -\frac{\partial}{\partial x_{n+1}}\left(\frac{\partial p}{\partial\tau}\right) = -\left(\frac{\partial f_1}{\partial x}\right)^T p - \left(\frac{\partial L}{\partial x}\right)^T - (x_{n+1} - t_0) \times \left(\left(\frac{\partial f_1}{\partial x}\right)^T \frac{\partial p}{\partial x_{n+1}} + \left(p^T \frac{\partial^2 f_1}{\partial x^2}\frac{\partial x}{\partial x_{n+1}}\right)^T + \left(p^T \frac{\partial^2 f_1}{\partial x\partial u}\frac{\partial u}{\partial x_{n+1}}\right)^T + \frac{\partial^2 L}{\partial x^2}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial x\partial u}\frac{\partial u}{\partial x_{n+1}}\right)$$

(1.2.26)

$$0 = \left(\frac{\partial f_1}{\partial u}\right)^T p + \left(\frac{\partial L}{\partial u}\right)^T + (x_{n+1} - t_0) \times \left(\left(\left(\frac{\partial f_1}{\partial x}\right)^T \frac{\partial p}{\partial x_{n+1}} + \left(p^T \frac{\partial^2 f_1}{\partial x^2}\frac{\partial x}{\partial x_{n+1}}\right)^T + \frac{\partial^2 L}{\partial x^2}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial x\partial u}\frac{\partial u}{\partial x_{n+1}}\right)\right)$$

(1.2.27)

for $\tau \in [0,1)$ and

$$\frac{\partial}{\partial\tau}\left(\frac{\partial x}{\partial x_{n+1}}\right) = \frac{\partial}{\partial x_{n+1}}\left(\frac{\partial x}{\partial\tau}\right) = -f_2 + \left(t_f - x_{n+1}\right) \times \left(\frac{\partial f_2}{\partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x_{n+1}}\right) \qquad (1.2.28)$$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p}{\partial x_{n+1}}\right) = -\frac{\partial}{\partial x_{n+1}}\left(\frac{\partial p}{\partial \tau}\right) = \left(\frac{\partial f_2}{\partial x}\right)^T p + \left(\frac{\partial L}{\partial x}\right)^T - (t_f - x_{n+1}) \times \left(\left(\frac{\partial f_2}{\partial x}\right)^T \frac{\partial p}{\partial x_{n+1}} + \right.$$
$$\left. + \left(p^T \frac{\partial^2 f_2}{\partial x^2}\frac{\partial x}{\partial x_{n+1}}\right)^T + \left(p^T \frac{\partial^2 f_2}{\partial u^2}\frac{\partial u}{\partial x_{n+1}}\right)^T + \frac{\partial^2 L}{\partial x^2}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial x \partial u}\frac{\partial u}{\partial x_{n+1}}\right) \quad (1.2.29)$$

$$0 = -\left(\frac{\partial f_2}{\partial u}\right)^T p - \left(\frac{\partial L}{\partial u}\right)^T + (t_f - x_{n+1}) \times \left(\frac{\partial f_2}{\partial u}\right)^T \frac{\partial p}{\partial x_{n+1}} + \left(p^T \frac{\partial^2 f_1}{\partial x \partial u}\frac{\partial u}{\partial x_{n+1}}\right)^T +$$
$$\left(p^T \frac{\partial^2 f_2}{\partial u^2}\frac{\partial u}{\partial x_{n+1}}\right)^T + \frac{\partial^2 L}{\partial x \partial u}\frac{\partial u}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial u^2}\frac{\partial u}{\partial x_{n+1}} \quad (1.2.30)$$

For $\tau \epsilon [1,2]$

Differentiating the boundary conditions (1.2.29) and (1.2.30) and the continuity condition (1.2.31) with respect to x_{n+1}, obtain that,

$$\frac{\partial x(0,x_{n+1})}{\partial x_{n+1}} = 0 \quad (1.2.31)$$

$$\frac{\partial p(2,x_{n+1})}{\partial x_{n+1}} = \frac{\partial^2 \psi(2,x_{n+1})}{\partial x^2}\frac{\partial x(2,x_{n+1})}{\partial x_{n+1}} \quad (1.2.32)$$

$$\frac{\partial p(1-,x_{n+1})}{\partial x_{n+1}} = \frac{\partial p(1+,x_{n+1})}{\partial x_{n+1}}$$

Problems with internally forced switchings

The specifications of switched system with IFS included the switching sets

$\Gamma_{(i_1,i_2)} \subseteq X_{i_1} \cap X_{i_2}$ where $X_i \epsilon R^n$. In thesis

$\Gamma_{(i_1,i_2)} = \left\{x | \gamma_{(i_1,i_2)}(x) = 0, \gamma_{(i_1,i_2)}: R^n \to R^{l_{(i_1,i_2)}}\right\}$.

Here we focus on optimal control roblems for switched systems withIFS in which a prespecified sequence of active subsystem is given.

Problem 4: Consider system with IFS. Given fixed time interval $[t_0, t_f]$ and prespecified sequence of active subsystem, find continuous input $u \epsilon U_{[t_0,t_f]}$ such that the coresponding continuos state trajectory x departs from a given initial state $x(t_0) = x_0$ and

$S_f = \{x | \phi_f(x) = 0, \phi_f: R^n \to R^{l_f}\}$. $J = \psi\left(\left(x(t_f)\right)\right) + \int_{t_0}^{t_f} L\left(x(t), u(t)\right) dt$

is minimized.

Problems of internally forced switching

Approach 1

1) Denote in redundant fashion that an optimal solution to an ıfs problem contaşns both optimal switching sequence and an potimal continuous input,i.e, regard an ıfs problem as an efs(externally forced switching) instants.

2) Verify the validitynof solution fort he ıfs problem if the system under the continuous input can evolve validly and generate the coresponding switching sequence.)

□

Theorem 2: Consider stage(a) for problem (4). Assume that subsystem k is active in $[t_{k-1,}t_k)$ for $1 \le k \le K$ and subsystem $K+1$ in $[t_{k-1,}t_k), t_{k+1}$=$t_{f.}$ Assume that

$x\epsilon\Gamma_k = \{x|\gamma_k(x) = 0 \; \gamma_k: R^n \to R^{l_k}\}$ at t_k. Let $u\epsilon U_{[t_0,t_f]}$ such that the coresponding continuos state trajectory x departs from a given initial state $x(t_0) = x_0$ and

$S_f = \{x|\phi_f(x) = 0, \phi_f: R^n \to R^{l_f}\}$.

Also assume that $x(t)\epsilon \; \imath nt(x_{K+1})$ for $t \in (t_{k-1}, t_k)$ $1 \le k \le K$ and $x(t) \in x_{K+1}$for

$t \in (t_K, t_f)$. In order to u be optimal,it is necessary that there exist vector function $p(t) = [p_1(t), \dots, p_n(t), t \in [t_0, t_f]$, such that conditions a)-c) as in theorem 1 hold, the condition hold.

d) At any t_k, k=1,2,…,K,we have $p(t_k -) + p(t_k -)$+$\left(\left(\frac{\partial\gamma_k}{\partial x}\right)(x(t_k)^T V_k\right)^T V_k = 0.$

Proof: Similar to theorem 1,except that here in J', we introduce a term $V_k^T\gamma_k(x(t_k))$ and in $\delta J'$,we have coefficients of $\delta x(t_k)$ as $p(t_k-)$+$\left(\left(\frac{\partial\gamma_k}{\partial x}\right)(x(t_k)^T V_k\right)^T$. Setting to zero coefficients of the independent increments of $\delta x(t_f), \delta x, \delta u, \delta p, \delta x(t_k)'s$ therefore yields the necessary conditions a)-d). □

2. SWITCHED OPTIMAL CONTROL FOR NONLINEAR OPTIMIZATION PROBLEM

2.1 Switched Systems

Definition:

$D(I,E)$ is directed graph indicating the discrete structure of system. The node set

$I = \{1,2,..,M\}$ is the set of indices for subsytems . The directed edge set E is a subset of $I \times I - \{(i,i|i \in I)\}$ which contains all valid events.If an event $e = (i_1, i_2)$ takes place,the system switches from subsystem i_1 to i_2. $F = \{f_i: R^n \times R^m \times R \to R^n, i \in I\}$

with f_i describing the vector field for the $i - th$ subsystem $\dot{x} = f_i(x,u,t)$,then the switched system can be define as,

$$\dot{x}(t) = f_{i(t)}(x(t), u(t), t) \quad (2.1.1)$$

$$i(t) = \varphi(x(t), i(t^-), t) \quad (2.1.2)$$

where $\varphi: R^n \times I \times R \to I$ determines the active subsystem at instant t.

Definition: For switched system S a switching sequence σ in $[t_0, t_f]$ is defined as

$$\sigma = \big((t_0, i_0), \dots, (t_k, i_k)\big) \quad (2.1.3)$$

with $0 \le K \le \infty, t_0 \le t_1 \le \cdots . t_K \le t_f$ $i_0 \in I$ $e_k = (i_{k-1}, i_k) \in E$

for $k = 1,2, \dots, K$. We define$\Sigma_{[t_0,t_f]} \triangleq \sigma's$ in$\big[t_0, t_f\big]$.

An optimal control problem

Problem 2.1 Consider a switched system S=(D,F).Given a fixed time interval$[t_0, t_f]$, find a picewise continuous input u and switching sequenceσ such that is minimized

$$\psi(x(t_f) + \int_{t_0}^{t_f} L(x(t), u(t))dt \quad (2.1.4)$$

Here $x(t_0) = x_0$

Problem 2.1 is basic optimal control problem in Bolza form. we assume that f,L are continuous and have continuous partial derivatives with respect the x, ϕ_f is continuos derivatives.

Two stage optimization

We need to find optimal control solution (σ^*, u^*) for Problem 2.1 such that

$$J(\sigma^*, u^*) = min_{\sigma\epsilon[t_0,t_f], u\epsilon U[t_0,t_f]} J(\sigma, u). \quad (2.1.5)$$

Note that for any fixed σ, Problem 2.1 reduces to conventional optimal problem which we need to find u that minimizes $J(\sigma^*, u^*)$= $J(\sigma, u)$. For this reason,

Lemma: For problem 2.1 if

a) an optimal solution (σ^*, u^*) exist and

b) for any fixed σ, there exist a corresponding $u^* = u_\sigma^*$ such that $J_\sigma(u) = J(\sigma, u)$ is minimized, then following equation holds,

$$min_{\sigma\epsilon\Sigma_{[t_0,t_f]}, u\epsilon U_{[t_0,t_f]}} J(\sigma, u) = min_{\sigma\epsilon\Sigma_{[t_0,t_f]}} min_{u\epsilon U_{[t_0,t_f]}} J(\sigma, u) \quad (2.1.6)$$

Proof: Firstly, $min_{\sigma\epsilon\Sigma_{[t_0,t_f]}, u\epsilon U_{[t_0,t_f]}} J(\sigma, u) \leq min_{\sigma\epsilon\Sigma_{[t_0,t_f]}, u\epsilon U_{[t_0,t_f]}} J(\sigma, u)$

(2.1.7)

Because for any fixed σ,there exist $u_\sigma^* \, u^*{}_\sigma$ such that $J(\sigma, u_\sigma^*) = min_{u\epsilon U_{[t_0,t_f]}} J(\sigma, u)$.But for every pair (σ, u_σ^*) we must have $J(\sigma^*, u^*) < J(\sigma, u_\sigma^*)$ therefore from (2.1.7) we must have

$$J(\sigma^*, u^*) \leq nf_{\sigma\epsilon\Sigma_{[t_0,t_f]}} = inf_{\sigma\epsilon\Sigma_{[t_0,t_f]}} min_{u\epsilon U_{[t_0,t_f]}} J(\sigma, u) \quad (2.1.8)$$

While we have inequality,

$$\inf_{\sigma\epsilon\Sigma_{[t_0,t_f]}} \min_{u\epsilon U_{[t_0,t_f]}} J(\sigma, u) \leq \min_{u\epsilon U_{[t_0,t_f]}} J(\sigma^*, u) = J(\sigma^*, u_{\sigma^*}^*)$$

(2.1.9)

we can choose $u_{\sigma^*}^* = u^*$, since for any other u ,we must have $J(\sigma^*, u^*)$ due to the optimality of (σ^*, u^*). Hence combining (2.1.8) and (2.1.9) we have

$$J(\sigma^*, u^*) \leq inf_{\sigma\epsilon\Sigma_{[t_0,t_f]}} min_{u\epsilon U_{[t_0,t_f]}} J(\sigma, u) \leq J(\sigma^*, u_{\sigma^*}^*) = J(\sigma^*, u^*) \quad (2.1.10)$$

Hence all inequalities in (2.1.10) must be inequalities and the $inf_{\sigma\epsilon\Sigma_{[t_0,t_f]}}$ can be replaced by $min_{\sigma\epsilon\Sigma_{[t_0,t_f]}}$, so we obtain that

$$J(\sigma^*, u^*) = min_{\epsilon\Sigma_{[t_0,t_f]}, u\epsilon U_{[t_0,t_f]}} J(\sigma, u) = min_{\sigma\epsilon\Sigma_{[t_0,t_f]}} min_{u\epsilon U_{[t_0,t_f]}}$$

(2.1.11)

Two stage optimization problem

Stage 1 . Fixing σ ,solve the inner minimization problem.

Stage 2. Regarding the optimal cost for each σas a function.

$$J_{1}=J_1(\sigma) = min_{u\epsilon U_{[t_0,t_f]}} J(\sigma,u) \quad (2.1.12)$$

Minimize J_1 with respect to $\sigma\epsilon\Sigma_{[t_0,t_f]}$.

Algorithm (A Two Stage Algorithm)

Stage 1 a) Fix the total numbwer of switchings to be K and the sequence of active subsystems and let the minimum value of J with respect to u be i.e, function of the K switching instants for $J_{1}=J_1(t_1,t_{2,\dots,}t_K)$ for K ≥0$(t_0,t_1,t_{2,\dots,}t_K \leq t_f)$. Find J_1.

b) Minimize it.

Stage 2 (a) Vary the sequence of active subsystems to find an optimal solution under K switchings.

(b) Vary the number of K switchings to find an optimal solution for problem 2.1.

More on stage 1 optimization

We concentrate of on stage 1 optimization. Stage 1(a) is in essence a conventional optimal control problem which seeks the minimum value of J with respect to u under given switching sequence $\sigma = \big((t_0,i_0),(t_1,e_1),\dots,(t_k,e_k)\big)$.

Stage 1(b) is in esence a constrained nonlinear optimization problem,

$$min_{\hat{t}}\, J_1(\hat{t}) \text{ subject to } t \,\epsilon\, T \quad (2.1.13)$$

where $T \triangleq \{\hat{t} = (t_1,\dots,t_K)^T | t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f\}$. In order to stage 1 problem,one needs to resort to not only optimal control methods, also nonlinear optimization techniques.

Stage 1(a)

For stage 1(a) where switching sequence $\sigma = \big((t_0,i_0),(t_1,e_1),\dots,(t_k,e_k)\big)$ is given , finding $J_1(\hat{t})$ for the coresponding $\hat{t} = (t_1,\dots,t_K)^T$ is a conventional optimal control problem. We need to find an optimal continuos input u and the coresponding J. In order to find solutions for stage 1(a) problems,computational methods must be adopted in most cases. Most of available numerical methods are for unconstrained conventional optimal control problems with fixed end time can be used.

Stage 1(b)

We need to solve the constrained nonlinear optimization problem (4.1) with simple constraints. Computational methods for the solution of such problems are abundant in the nonlinear optimization literature.

Algorithm (A conceptual Algorithm For Stage 1 Optimization)

(1) Set the iteration index $J = 0$. Choose an initial $\hat{t}^j$

(2) By solving an optimal control problem stage1 (a), find $J_1(\hat{t}^j)$.

(3) Find $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 j_1}{\partial \hat{t}^2}(\hat{t}^j)$.

(4) Use some feasible direction method to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$. Set the iteration index $j = j + 1$.

(5) Repeat steps (2),(3),(4) and (5), until prespecified termination condition is satisfied.

□

Key elements of the above the algorithm are

(a) An optimal control for Step (2).
(b) The derivations of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 j_1}{\partial \hat{t}^2}(\hat{t}^j)$ for step (3).
(c) A nonlinear optimization step algorithm for step (4).

Optimization for stage 1 problem based on direct differentiations

We propose a method to approximate the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 j_1}{\partial \hat{t}^2}(\hat{t}^j)$ and which can be used in stage1(b)optimizations. The method is based on direct differentiations of the value functions. We have assume that u is piecewise differentiable. We need to find an optimal switching instant vector $\hat{t} = (t_1, t_2, \ldots, t_K)$ and optimal control input u.

Assume that we have nominal $\hat{t} = (t_1, t_2, \ldots, t_K)$and nominal control input u which is piecewise smooth. If they are both fixed,then the cost J will be function of$(x(t_0), t_0)$, but if u is fixed and $\hat{t}$ can be varied in small neighborhood of nominal value,then cost J will be function of $(x(t_0), t_0, t_1, \ldots, t_K)$.

Now let us assume that along with the small variations of $\hat{t}, u$ varies correspondingly in the following manner. If varies to $\hat{t} + d\hat{t}, u$ varies correspondingly to

$$\begin{cases} u(t_k -) + (t - t_k)\dot{u}^{k-}, \text{ if } t \in [t_k, t_k + dt_k) \text{ for } dt_k \geq 0 \\ u(t_k +) + (t - t_k)\dot{u}^{k+}, \text{ if } [t_k + dt_k, t_k] \text{ for } dt_k < 0 \\ u(t), \qquad , \text{else} \end{cases} \qquad (2.1.14)$$

Where $\dot{u}^{k-} \triangleq \frac{du(t_k-)}{dt}$ and $\dot{u}^{k+} \triangleq \frac{du(t_k+)}{dt}$. We say that u assumes open loop variations in this case means that $u(t)$ only has variations in the interval between t_k and $t_k + dt_k$ as shown in figure 1. We denote such a cost value function(which is not necessarily optimal)

$$V^0(x(t_0), t_1, \dots, t_K) = \psi\left(x(t_f)\right) + \int_{t_0}^{t_1} L(x, u, t)dt + \int_{t_K}^{t_f} L(x, u, t)dt \qquad (2.1.15)$$

where the superscript 0 is to indicate that the starting time for evaluation is t_0. We can define the value of function at the k th switching instant as

$$V^K(x(t_0), t_1, \dots, t_K) = \psi\left(x(t_f)\right) + \int_{t_K}^{t_{K+1}} L(x, u, t)k\, dt + \int_{t_K}^{t_f} L(x, u, t)dt$$

(2.1.16)

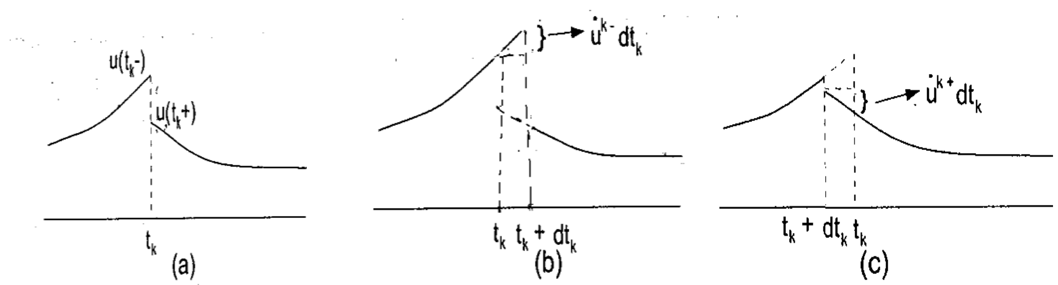

Figure 1: The solid curves are $u(t)$. (a) The nominal input $u(t)$. (b) The open loop variations of $u(t)$ induced by $dt_k \geq 0$. (c) The open loop variations of $u(t)$ induced by $dt_k < 0$

Single switching

Assume that we are given nominal t_1, a nominal u and the coresponding nominal state trajectory x. We denote $\hat{u}(t)$and $\hat{x}$(t)o be input and state trajectory after variation dt_1has taken place. We can write function with a superscript 1-(resp 1+) whenever it is evaluated at t_1 and the nominal values

$x(t_1), u(t_1 -)$, resp. t_1 and the nominal values $x(t_1), u(t_1 -)$. Examples of this notational convention are

$f^{1-} = f_1(x(t_1), u(t_1 +), t_1)$, $f^{1+} = f_2(x(t_1), u(t_1 +), t_1)$,
$L^{1-} = L(x(t_1), u(t_1 -), t_1)$

$L^{1+} = L(x(t_1), u(t_1 +), t_1)$, $V^{1+} = V^1(x(t_1), t_1)$.

It is not diffucult to see that $V^0(x_0, t_0, t_1) = V^1(x(t_1), t_1) + \int_{t_0}^{t_1} L(x, u, t)dt$ (2.1.17)

For a small variationdt_1 of t_1, we have

$$V^0(x_0, t_0, t_1 + dt_1) = V^1(\hat{x}(t_1 + dt_1), t_1 + dt_1) + \int_{t_0}^{t_1+dt_1} L(x, u, t)dt \qquad (2.1.18)$$

The first term in (2.1.18) can be expanded into second order as

$$V^1(\hat{x}(t_1 + dt_1), t_1 + dt_1) = V^{1+} + V_x^{1+}dx(t_1) + V_{t_1}^{1+}dt_1 + \frac{1}{2}(dx(t_1))^T V_{xx}^{1+}dx(t_1) + \frac{1}{2}V_{t_1t_1}^{1+}dt_1 \qquad (2.1.19)$$

where $dx(t_1) \triangleq \hat{x}(t_1 + dt_1) - x(t_1) = f^{1-}dt_1 + \frac{1}{2}(f_t^{1-} + f_x^{1-}f_1^{-} + fu^{1-}\dot{u}^{1-})dt_1{}^2 + o(dt_1{}^2)$ (2.1.20)

The second order expansion of the second term is derived as follows by distinguishing the case of $dt_1 \geq 0$.

If $dt_1 \geq 0$, we have

$$\int_{t_0}^{t_1+dt_1} L(\hat{x}, \hat{u}, t) = \int_{t_0}^{t_1} L(x, u, t)dt + \int_{t_1}^{t_1+dt_1} L(x, u, t)dt$$

$$= \int_{t_0}^{t_1} L(x, u, t)dt + L^{1-}dt_1 + \frac{1}{2}dt_1 L_x^{1-}dx(t_1) + \frac{1}{2}dt_1 L_u^{1-}du(t_1) + \frac{1}{2}L_t^{1-}d(t_1)^2 \qquad (2.1.21)$$

if $dt_1 < 0$,we have $\int_{t_0}^{t_1+dt_1} L(\hat{x}, \hat{u}, t) = \int_{t_0}^{t_1} L(x, u, t)dt + \int_{t_1}^{t_1+dt_1} L(x, u, t)dt$ =

$$\int_{t_0}^{t_1} L(x, u, t)dt + L^{1-}dt_1 + \frac{1}{2}dt_1 L_x^{1-}dx(t_1) + \frac{1}{2}dt_1 L_u^{1-}du(t_1) + \frac{1}{2}L_t^{1-} + d(t_1)^2 \qquad (2.1.22)$$

which has same expression as (2.1.21) for $dt_1 \geq 0$ although the derivation is slightly different. Note that ,

$$du(t_1) \triangleq \hat{u}\big((t_1 + dt_1) - u(t_1 -)\big) = \begin{cases} \dot{u}^{1+}dt_1 & \text{for } dt_1 \geq 0. \\ \dot{u}^{1-}dt_1 + o(dt_1) & \text{for } dt_1 < 0 \end{cases} \qquad (2.1.23)$$

$$V^0(x_0, t_0, t_1) = V^{1+} + \int_{t_0}^{t_1} L(x,u,t)dt + V_x^{1+}\, dx(t_1) + V_{t_1}^{1+} dt_1 + L^{1-} dt_1 + \frac{1}{2}(dx(t_1))^T V_{xx}^{1+} dx(t_1) + \frac{1}{2} V_{t_1 t_1}^{1+} {dt_1}^2 + dt_1 V_{t_1 x}^{1+} dx(t_1) + \frac{1}{2} dt_1 L_x^{1-} dx(t_1) + \frac{1}{2} dt_1 L_u^{1-} du(t_1) + \frac{1}{2} L_t^{1-} {dt_1}^2 \quad (2.1.24)$$

$$= V^0(x_0, t_0, t_1) + \left(V_x^{1+} f^{1-} + V_{t_1}^{1+} + L^{1-}\right) dt_1 + \frac{1}{2}\Big[V_x^{1+} (f^{1-} + f_x^{1-} f^{1-} + f_u^{1-} \dot{u}^{1-} + (f^{1-})^t V_{xx}^{1+} f^{1-} + V_{t_1 t_1}^{1+} + 2V_{t_1 x}^{1+} f^{1-} + L_x^{1-} f^{1-} + L_u^{1-} \dot{u}^{1-} + L_t^{1-} \Big] {dt_1}^2 \text{ for all } dt_1.$$

(2.1.25)

Consider V^{1+} is value function for given nominal value $u(t)$.

We have $V_{t_1}^{1+} = -V_x^{1+} f^{1+} - L^{1+}$ (2.1.26)

By differentiating (2.1.25) , we obtain

$$V_{t_{1x}}^{1+} = -V_{xx}^{1+}(f^{1+}) - V_x^{1+} f_x^{1+} - L_x^{1+}$$

$$V_{t_{1}t_1}^{1+} = -V_{t_{1x}}^{1+} f^{1+} - V_x^{1+} f_t^{1+} - L_t^{1+} - (V_x^{1+} f_u^{1+} - L_u^{1+})\dot{u}^{1+}$$

$$V_{t_1 t_1}^{1+} = (f^{1+})^T V_{xx}^{1+} f^{1+} + (V_x^{1+} f_X^{1+} + L_X^{1+}) f^{1+} + V_x^{1+} f_t^{1+} - L_t^{1+} + (V_x^{1+} f_u^{1+} + L_u^{1+})\dot{u}^{1+} \quad (2.1.27)$$

By substuting (2.1.25), (2.1.26), (2.1.27) we can write

$$V_{t_1}^0 = L^{1-} - L^{1+} + V_x^{1+}(f^{1-} - f^{1-}) \quad (2.1.28)$$

$$V_{t_{1}t_1}^0 = L^{1-} - L^{1+} + V_{xx}^{1+}(f^{1-} - f^{1-})^T(f^{1-} - f^{1-}) - (V_x^{1+} f_x^{1+} + L_x^{1+})(f^{1-} - f^{1-})$$

$$+(V_x^{1+}\left({f_x}^{1-} - {f_x}^{1+}\right) + V_x^{1-} - L_x^{1-}) f^{1-} + (V_x^{1+}\left({f_x}^{1-} - {f_x}^{1+}\right) - {L_t}^{1-} - {L_t}^{1+} + (V_x^{1+} f_u^{1-} + L_u^{1-})\dot{u}^{1-} - (V_x^{1+} f_u^{1+} + L_u^{1+})\dot{u}^{1-}$$

(2.1.29)

2.2 Two or more switching

From second order optimization algorithm ,for switched system two or more switchings,we neeed more information to derives of V^0 with respect to t_k 's. Let us first consider case of two switchings. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 ($t_0 \leq t_1 \leq \cdots . t_f$). The value function is then,

$(t_0 \le t_1 \le \cdots . t_f)$. The value function is then,

$$V^0(x_0, t_0, t_{1,}t_2) = V^1(x(t_1), t_1) + \int_{t_o}^{t_1} L(x,u,t)dt \qquad (2.2.1)$$

$$= V^2(x(t_2), t_2) + \int_{t_o}^{t_2} L(x,u,t)dt \qquad (2.2.2)$$

Definition (Incremental change) : Given any variations dt_1 and dt_2, we define

$\delta x(t), min\{t_1 + dt_1\} \le t \le max\{t_2 + dt_2\}$ to be incremental change of the state due to dt_1 and dt_2. In detail see figure 3.

Case 1: $dt_1 \ge 0, dt_2 \ge 0$, (see figure 3(a)).In this case $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x} - x(t), t \in [t_1 + dt_1, t_2] \\ y_1(t) - x(t), t \in [t_1, t_1 + dt_1] \\ \hat{x}(t) - x(t), t \in [t_2, t_2 + dt_1,] \end{cases} \qquad (2.2.3)$$

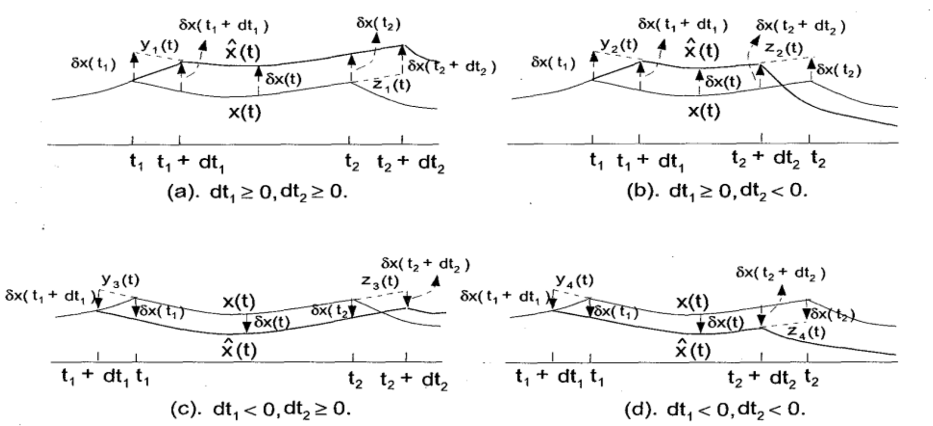

Figure 2: The incremental change $\delta x(t)$ for (a), where $y_1(t)$ is solution of

$$\begin{cases} \dot{y}_1(t) = f_2(y_1(t), u(t), t)\ , t \in [t_1, t_1 + dt_1] \\ y_1(t_1 + dt_1) = \hat{x}_1(t_1 + dt_1) \end{cases} \qquad (2.2.4)$$

And $z_1(t)$ is solution of

$$\begin{cases} \dot{z}_1(t) = f_2(z_1(t), \hat{u}(t), t)\ , t \in [t_2, t_2 + dt_2] \\ z_1(t_2) = x(t_{2)} \end{cases} \qquad (2.2.5)$$

Case2: $dt_1 \geq 0$, $dt_2 < 0$ (see figure 3(b).)

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), t \in [t_1 + dt_1, t_2 + dt_2] \\ y_2(t) - x(t), t \in [t_1, t_1 + dt_1] \\ z_2(t) - x(t), t \in [t_2, t_2 + dt_2] \end{cases} \quad (2.2.6)$$

$y_2(t)$ is solution of

$$\begin{cases} \dot{y}_2(t) = f_2(y_2(t), u(t), t)\ , t \in [t_1, t_1 + dt_1] \\ y_2(t_1 + dt_1) = \hat{x}(t_1 + dt_1) \end{cases} \quad (2.2.7)$$

And $z_2(t)$ is solution of

$$\begin{cases} \dot{z}_2(t) = f_2(z_2(t), u(t), t)\ , t \in [t_2, t_1 + dt_2] \\ z_2(t_2 + dt_2) = \hat{x}(t_2 + dt_2) \end{cases} \quad (2.2.8)$$

Case 3: $dt_1 < 0, dt_2 \geq 0$ (see figure 3(c)).

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), t \in [t_1, t_2] \\ \hat{x}(t) - y_3(t), t \in [t_1 + dt_1, t_1] \\ \hat{x}(t) - z_3(t), t \in [t_2, t_1 + dt_2] \end{cases} \quad (2.2.9)$$

Where $y_3(t)$ is solution of

$$\begin{cases} \dot{y}_3(t) = f_2(y_3(t), \hat{u}(t), t)\ , t \in [t_1, t_1 + dt_1] \\ y_3(t_1) = x(t_1) \end{cases} \quad (2.2.10)$$

And $z_3(t)$ is solution of

$$\begin{cases} \dot{z}_3(t) = f_2(z_3(t), \hat{u}(t), t)\ , t \in [t_2, t_1 + dt_2] \\ z_3(t_2) = x(t_2) \end{cases} \quad (2.2.11)$$

Case 4: $dt_1 < 0, dt_2 < 0$ see figure 3(d).

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), t \in [t_1, t_2 + dt_2] \\ \hat{x}(t) - y_4(t), t \in [t_1, t_1 + dt_1] \\ z_4(t) - x(t), t \in [t_2 + dt_2, t_2] \end{cases} \quad (2.2.12)$$

where $y_4(t)$ is solution of
$$\begin{cases} \dot{y}_4(t) = f_2(y_4(t), \hat{u}(t), t), t \in [t_1, t_1 + dt_1] \\ y_4(t_1) = x(t_1) \end{cases} \quad (2.2.13)$$

And $z_4(t)$ is solution of
$$\begin{cases} \dot{z}_4(t) = f_2(z_4(t), u(t), t), t \in [t_2, t_2 + d, t_2] \\ z_4(t_2 + dt_2) = \hat{x}(t_2 + dt_2) \end{cases} \quad (2.2.14)$$

Note that $\delta x(t)$ defines the between $x(t)$ and $\hat{x}(t)$ in the time interval where subsystem 2 is active.

Lemma 2.2.1: Let $g(t,u)$ be areal continuous function of pair of variables $t \in (a,b)$, $,u \in U \subseteq R^m$ and let $u(t)$, $a < t < b$ be a piecewise continuos function with values in U. If θ is point in(a,b) at one of the following three conditions satisfied

(a) θ is a point at which u is continuous and p, q are arbitrary real numbers,

(b) θ is a point at which u is continuous and p, q are positive,

(c) θ is a point at which u is continuous and p, q are negative ,then we have

$$\int_{\theta+p\varepsilon}^{\theta+q\varepsilon} g(t, u(t))dt = \varepsilon(q-p)g(\theta, u(\theta)) + o(\varepsilon) \quad (2.2.15)$$

Here ε is sufficiently small positive number and $o(\varepsilon)$ is an finite small of higher order than ε, i.e $\underbrace{\lim}_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon}$=0.

Lemma 2.2.2: The expressions of $\delta x(t_2)$ and $\delta x(t_2 + dt_2)$ are as follows

$$\delta x(t_2) = A(t_2, t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1)\ \delta x(t_2 + dt_2) \quad (2.2.16)$$

$$\delta x(t_2 + dt_2) = A(t_2, t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-} A(t_2, t_1)(f^{1-} - f^{1+})dt_1 dt_{2.} \quad (2.2.17)$$

Where $A(t_2, t_1)$ is transion matrix for variational time varying equation

$$\dot{y}(t) = \frac{\partial f(x(t),u(t),t)}{\partial x} y(t) \qquad (2.2.18)$$

for $y(t)$ $t \in [t_1, t_2]$ in (2.2.18) f is coresponding active subsystem vector field in $[t_1, t_2]$ and u, x are current nominal input and state.

The forward decoupling principle: If u assumes open loop variations ,then

(a) The value of incremental change $\delta x(t_1)$ at t_1 will not be dependent upon dt_2.

(b) The value of incremental change $\delta x(t_2)$ at t_2 will be dependent upon dt_2.

Lemma 2.2.3:The expression of dxt_2 is

$$dxt_2 = A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1 dt_2 + f^{2-}dt_2 + \text{other terms} \qquad (2.2.19)$$

Proof: proof is directly from the fact that

$$dxt_2 = \delta x(t_2 + dt_2) + f_2(x(t_2), u(t_2 -), t_2)dt_2 + o(dt_2) \qquad (2.2.20)$$

Remark: It is important that dxt_2 ,we delibrately express the term ,

$f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1 dt_2$ explicitly because it will contribute to the coefficient $dt_1 dt_2$ as can be seen from the discussions below. We have expressions $\delta x(t_2), \delta x(t_2 + dt_2)$ and $x(t_2)$ we are ready to derive the coefficient for $dt_1 dt_2$ in expansion of

$$V^0(x_0, t_0, t_1 + dt_1, t_2 + dt_2) = V^2(\hat{x}(t_2 + dt_2), t_2 + dt_2) + \int_{t_2}^{t_2+dt_2} L(\hat{x}(t), u(t), t)dt \qquad (2.2.21)$$

Taylor expansion of first term is

$$V^2(\hat{x}(t_2 + dt_2), t_2 + dt_2 = V^{2+} + V_x^{2+}dx(t_2) + V_{t_2}^{2+}dx(t_2) + \frac{1}{2}(dx(t_2))^T V_{xx}^{2+}dx(t_2) + \frac{1}{2}V_{t_2 t_2}^{2+}dx(t_2) + o(dt_2)^2 \qquad (2.2.22)$$

In (2.2.21) the terms that will possibly contribute to coefficient $dt_1 dt_2$ are those containing $dx(t_2)$. They are $V_x^{2+}dx(t_2), \frac{1}{2}(dx(t_2))^T V_{xx}^{2+}dx(t_2), d(t_2), V_{t_2 x}^{2+}dx(t_2)$.

(2.2.23)

Substituting $dx(t_2)$ into (2.2.22) and summing them, we have first term to the coefficient of $dt_1 dt_2$ as $[V_x^{2+}f_x^{2-} + (f^{2-})^T V_{xx}^{2+} + V_{t_2 x}^{2+}]A(t_2,t_1), (f^{1-} - f^{1+})$ (2.2.24)

For the second term in (2.2.21) we have lemma.

Lemma 2.2.4: The contribution of $\int_{t_0}^{t_2} L(\hat{x}(t), \hat{u}(t), t)dt$ to the coefficient of $dt_1 dt_2$ is $L_x^{2-} A(t_2, t_1), (f^{1-} - f^{1+})$. (2.2.25)

Remark : The above this results stil holds even when $t_2 = t_1$. $V_{t_2 x}^{2+}$ which can be obtained similarly to $V_{t_1 x}^{1+}$ finally we have

$$V_{t_1 t_2}^0 = [V_x^{2+} + f_x^{2-} + (f^{2-})^T V_{xx}^{2+} + V_{t_2 x}^{2+} + L_x^{2-}] A(t_2, t_1), (f^{1-} - f^{1+})$$

$$= [V_x^{2+} + (f_x^{2-} - f_x^{2+}) + (f_x^{2-} - f_x^{2+})^T V_{xx}^{2+} + L_x^{2-} - L_x^{2+}] A(t_2, t_1), (f^{1-} - f^{1+})$$

(2.2.26)

This result can be extended to case of K switchings to relate $\delta x(t_1)$ and dt_k .

The implementation of algorithm

This algorithm is modified version of the conceptual Algorithm 4.1 can be used for Stage 1 optimization.

Algorithm (Algorithm for stage 1 optimization)

(1) Set the iteration index $j = 0$ .Choose an initial $\hat{t}^j$.

(2) By solving an optimal control problem fort he curren$\hat{t}^j$t (Stage(1)a),find the coresponding optimal or suboptimal control input u^j .

(3) For the currrent $\hat{t}^j$and its coresponding u^j ,supposing that u^j assumes poen loop variations, find $\frac{\partial V^0}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 V^0}{\partial \hat{t}^2}(\hat{t}^j)$ as approximations to $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 j_1}{\partial \hat{t}^2}(\hat{t}^j)$

(4) Use some faesible direction method to update to be$\hat{t}^j \leq 0, \hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$. Set the iteration index $j = j + 1$.

Proof of Lemma 2.2.5:

Case1: $dt_1 \geq 0, dt_2 \geq 0$.

$$\delta x(t_1 + dt_1) = \int_{t_1}^{t_1 + dt_1} f_1(\hat{x}, \hat{u}, t)dt - \int_{t_1}^{t_1 + dt_1} f_2(\hat{x}, \hat{u}, t)dt$$

Using lemma 2.2.1 ,

$$\int_{t_1}^{t_1 + dt_1} f_1(\hat{x}, \hat{u}, t)dt = f_1(\hat{x}(t_1), \hat{u}(t_1), t_1)dt_1 + o(dt_1)$$

$$\delta x(t_1+dt_1)=\int_{t_1}^{t_1+dt_1} f_1(\hat{x},\hat{u},t)dt-\int_{t_1}^{t_1+dt_1} f_2(\hat{x},\hat{u},t)dt$$

Using lemma 2.2.1,

$$\int_{t_1}^{t_1+dt_1} f_1(\hat{x},\hat{u},t)dt = f_1(\hat{x}(t_1),\hat{u}(t_1),t_1)dt_1+o(dt_1)$$
$$= f_1(x(t_1),u(t_1-),t_1)dt_1+o(dt_1)$$
$$= f^{1-}dt_1+o(dt_1)$$

Using lemma 2.2.1, $(x(t_1),u(t_1),t_1)dt_1+o(dt_1$

$$\int_{t_1}^{t_1+dt_1} f_2(x,u,t)dt = f_2(x(t_1),u(t_1+),t_1)dt_1+o(dt_1)$$
$$= f^{1-}dt_1+o(dt_1)$$

Hence $\delta x(t_1+dt_1)=(f^{1-}-f^{1+})dt_1+o(dt_1)$ and we conculude that form property ofvariational equation that,

$$\delta x(t_2)=A(t_2,t_1+dt_1)\delta x(t_1+dt_1)+o(dt_1)$$
$$=[A(t_2,t_1)+At_1dt_1+o(dt_1)](f^{1-}-f^{1+})dt_1+o(dt_1))+o(dt_1)$$
$$=A(t_2,t_1)(f^{1-}-f^{1+})dt_1+o(dt_1)$$
$$\delta x(t_2+dt_2)=$$
$$\hat{x}(t_2)+\int_{t_2}^{t_2+dt_2} f_2(\hat{x},\hat{u},t)dt-z_1(t_2)+\int_{t_2}^{t_2+dt_2} f_2(z_1(t),\hat{u},t)dt$$
$$\delta x(t_2+dt_2)=\delta x(t_2)+\int_{t_2}^{t_2+dt_2} f_2(\hat{x},\hat{u},t)dt-\int_{t_2}^{t_2+dt_2}(z_1(t),\hat{u},t)dt$$
$$\delta x(t_2+dt_2)=\delta x(t_2)+f_2(\hat{x}(t_2),u(t_2-),t_2)-f_2(x(t_2),u(t_2-),t_2)dt_2+o(dt_1)$$
$$=\delta x(t_2)+f_x^{2-}\delta x(t_2)dt_2+o(dt_2)$$
$$=A(t_2,t_1)(f^{1-}-f^{1+})dt_1+f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2+$$

other terms

Case 2: $dt_1\geq 0, dt_2<0$,

$$\delta x(t_2+dt_2)=z_2(t_2)+\int_{t_2}^{t_2+dt_2} f_2(z_2(t_2),u,t)dt-x(t_2)+$$
$$\int_{t_2}^{t_2+dt_2} f_2(x(t),u,t)dt=z_2(t_2+dt_2)-x(t_2+dt_2)$$

$= z_2(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_2(t_2)\,, u, t)dt - x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(x(t)\,, u, t)dt =$

$z_2(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_2(t_2)\,, u, t)dt - x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(x(t)\,, u, t)dt$

$= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_2(t_2)\,, u, t)dt - f_2(x(t), u(t), t)dt$

$= \delta x(t_2) + f_2(z_2(t_2), u(t_2 -), t_2)dt_2 - f_2(x(t_2), u(t_2 -), t_2)dt_2$

$+o(dt_2)$

$= \delta x(t_2)+f_x^{2-}\delta x(t_2)dt_2 + o(dt_2)$

$= A(t_2, t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2, t_1)(f^{1-} - f^{1+})dt_1 dt_2 +$

other terms

Case 3: $dt_1 < 0, dt_2 \geq 0$

$\delta x(t_1) = \int_{t_1}^{t_1+dt_1} f_2(\hat{x}\,(t), \hat{u}(t), t)dt - \int_{t_1}^{t_1+dt_1} f_1(x\,(t), u(t), t)dt$

$= f_2(x(t_1 + dt_1), u(t_1 +), \dot{u}^{1+}dt_1, t_1 + dt_1)(-dt_1) - f_1(x(t_1 + dt_1),$

$u(t_1 + dt_1), (t_1 + dt_1)(-dt_1) + o(dt_1)$

$= f_1(x(t_1), u(t_1 -), t_1)dt_1 - f_2(x(t_1), u(t_1 +), t_1)dt_1 + o(dt_1)$

$= (f^{1-} - f^{1+})dt_1 + o(dt_1)$

we use last equations,

$x(t_1 + dt_1) = x(t_1) + \dot{x}\,(t_1 -)dt_1 + o(dt_1),$

$u(t_1 + dt_1) = u(t_{1-}) + \dot{u}\,(t_1 -)dt_1 + o(dt_1)$

And

$\delta x(t_2) = A(t_2, t_1)\delta x(t_1) + o(dt_1) + [\hat{x}(t_2) + \int_{t_2}^{t_2+dt_2} f_2(\hat{x}\,(t), \hat{u}(t), t)dt] -$

$z_3(t_2)$

$= A(t_2, t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1)$

$= [\hat{x}(t_2) + \int_{t_2}^{t_2+dt_2} f_2(\hat{x}\,(t), \hat{u}(t), t)dt] - z_3(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_3\,(t), \hat{u}(t), t)dt$

$= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(\hat{x}\,(t), \hat{u}(t), t)dt - \int_{t_2}^{t_2+dt_2} f_2(z_3\,(t), \hat{u}(t), t)dt$

$= \delta x(t_2) + f_2(\hat{x}(t_2), u(t_2 -), t_2)dt_2 - f_2(x(t_2), u(t_2 -), t_2)dt_2 + o(dt_2)$

$= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2)$

Case 4: $dt_1 < 0, dt_2 < 0$

$\delta x(t_2 + dt_2) =$

$[z_4(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_4(t), u(t), t)dt] - [x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(x(t), u(t), t)dt]$

$$= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_4(t), u(t), t)dt - f_2(x), u(t), t]dt$$

$$= \delta x(t_2) + f_2(z_4, u(t_2 -), t_2) - f_2(x(t_2), u(t_2 -), t_2)]dt_2 + o(dt_2)$$
$$= \delta x(t_2) + f_x^{2-} \delta x(t_2) dt_2 + o(dt_2)$$

Proof for Lemma 5.4: Note that

$$\int_{t_0}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)dt$$
$$= \int_{0}^{\max\{t_1, t_1+dt_1\}} L(\hat{x}, \hat{u}, t)dt + \int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(x + \delta x, \hat{u}, t)d$$

Case 1: $\int_{t_0}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)dt$

$\int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)dt =$

$\int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(x + \delta x, \hat{u}, t)dt + \int_{t_2}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)dt$

$\delta x(t) = A(t, t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1),\ \hat{u}(t) = u(t)$

$\int_{t_2}^{t_2+dt_2} L(\hat{x}(t_2), \hat{u}(t), t)dt = L(\hat{x}(t_2), u(t_2 -)dt_2 + o(dt_2)$

$$= L(x(t_2), u(t_2 -), t_2)dt_2 + L_x^{2-} \delta x(t_2)dt_2$$

Case 2: $dt_2 < 0, x(t) = \delta x(t) = \hat{x}(t)$ and $\hat{u}(t) = u(t)$

$\int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)dt = \int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(x + \delta x, u, t)dt =$

$\int_{\max\{t_1, t_1+dt_1\}}^{t_2} L(x + \delta x, u, t)dt + \int_{t_2}^{t_2+dt_2} L(x + \delta x, u, t)dt$

$\int_{t_2}^{t_2+dt_2} L(x + \delta x, u, t)dt = L(x(t_2) + \delta x(t_2), u(t_2 -), t_2)dt_2 + o(dt_2) =$

$L(x(t_2), u(t_2 -), t_2)dt_2 + L_x^{2-} \delta x(t_2)dt_2.$

3. TIME DELAY OPTIMAL CONTROL PROBLEM

3.1 Problem Formulation

Consider switched dynamical systems defined in $[0, T]$ with one time delay and N-1 switches:

$$\dot{x}(t) = f_i\big(t, x(t), x(t-h)\big), \quad t \,\epsilon \big(\tau_{i-1,} \tau_i\big], \quad i = 1,2, \dots, N \tag{3.1.a}$$

With initial condition $x(0) = x_0, \quad x(t) = \phi(t), \quad t\epsilon[-h, 0),$ (3.1.b)

Where $x \in R^n$, h is delay time, $f_i: R^{n+n+1} \to R^n, i = 1, \dots., N$ and $\phi: R^1 \to R^n$ are given functions. Assume that the switching sequence is preassigned, such as

$$0 = \tau_0 \leq \tau_1 \leq \cdots . \leq \tau_{N-1} \leq \tau_N = T \tag{3.1.2}$$

where the switching times $\tau_i, i = 1, \dots \dots N-1$, are decision variables. This approach is to find a switching vector $\tau = (\tau_1, \tau_2, \dots \dots, \tau_{N-1})$ subject to condition (2.2) for time delayed switched systems(2.1a) and (2.1b) such cost function $J(\tau) = \Phi(x(T|\tau))$

(3.1.3)

is minimized, where $x(T|\tau)$ is solutiuon of system (3.1a) and (3.1b) at terminal time $t = T$ corresponding to the switching vector $\tau = (\tau_1, \dots., \tau_{N-1})$.

Remark

If the cost function is given by

$$J = (\tau) = \Phi(x(T|\tau)) + \int_0^T \mathcal{L}(t, x(t|\tau), x(t-h|\tau))dt, \tag{3.1.4}$$

convert it into an objective function of the form (2.3) by introducing an additional state with dynamics

$$\dot{x}_{n+1}(t) = \mathcal{L}\big(t, x(t|\tau), x(t-h|\tau)\big), \qquad x_{n+1}(0) = 0.$$

The objective function of (2.4) can be written as

$J(\tau) = \widehat{\Phi}\big(\hat{x}(T|\tau)\big)$, where $\hat{x}(T|\tau) = [x(T|\tau)^T, x_{n+1}(T|\tau)]^T$ and

$$\widehat{\Phi}\big(\hat{x}(T|\tau)\big) = \Phi\big(x(T|\tau)\big) + x_{n+1}(T|\tau).$$

We assume that the following conditions are satisfied:

(1) all switching durations are larger than the delay time h, $i.e$,

$$\tau_i - \tau_{i-1} \geq h, \quad \forall i = 1,2, \dots., N \tag{3.1.5}$$

(2) the functions f_i "$(t, x(t), x(t-h)), i = 1,2, \dots., N$" and $\Phi\big(x(\mathrm{T})\big)$ are continuously differentiable.

3.2 Problem Formulation and Gradient Formula

To solve this problem we need gradient Formula of terminal cost function with respect to switching vector τ.

For each $i = 1, \dots, N$, ξ_i=τ_i-$\tau_{i-1,}$ i=1,…,N (3.2.1)
be duration between the switching times τ_{l-1} and τ_i .Clearly that,

$\tau_i = \sum_{j=1}^{i} \xi_{j,}$ i=1,…,N (3.2.2)
Let $\xi = (\xi_{1,\dots,}\xi_n)\epsilon R^n$ be duration vector.

$\xi_i \geq 0,$ i=1,…,N (3.2.3)

$\sum_{l=1}^{N} \xi_i = T$ (3.2.4)

The determination of switching vector is equivalent to determination of duration vector. Also, $x(t)$, which is dependent only on switching instants $\{\tau_i: \tau_i \leq t, i = 1, .., N\}$, can be viewed as being dependent on duration vector ,i.e,

$x(t) = x(t; \xi_1, \dots., \xi_{i-1}$,for $t\epsilon\left(\tau_{i-1,}\tau_i\right], i = 1, \dots, N$. Then,(2.1a),(2.1b) we can write

$\frac{\partial x}{\partial t}(t; \xi_{i-1}, \xi_{i-2}, \dots., \xi_1) = f_i(t, x\left(t, \xi_{i-1,}\xi_{i-2}, \dots, \xi_1\right), x(t-h, \xi_{i-1}, \xi_{i-2}, \dots, \xi_1))$

$t\epsilon(\tau_{i-1}, \tau_i],\ i = 1, \dots. N$ (3.2.5a)
With

$x(t, \xi_{i-1}, \xi_{i-2}, \dots, \xi_1)|_{t=\tau_{i-1}} = x(t, \xi_{i-2}, \dots,)|_{t=\tau_{i-1}}$ (3.2.5b)

$x(t-h, \xi_{i-1}$,…,$\xi_1) = x\left(\tau_{i-1} + t - h; \xi_{i-2,\dots,}\xi_1\right),$ (3.2.5c)

For $t\epsilon(\tau_{i-1}, \tau_{i-1} + h], i = 2, \dots, N$ and

$x(t)|_{t=0} = x_0$ (3.2.5d)

$x(t) = \Phi(t),$ $t\epsilon\ [-h, 0]$ (3.2.5e)

And $J(\xi) = \Phi\left(x(T|\xi)\right),$ (3.2.6)

Which $x(.|\xi)$is solution of (3.5).

Now this problem can be formulated as;

Given dynamical system (3.2.5) find a duration vector $\xi\epsilon R^N$satisfying (3.2.3) and (3.2.4) such that the terminal cost function (3.6) is minimized.This problem referred to as problem(RP).To solve(RP),we need the gradients of terminal cost (3.6) with respsect to duration vector ξ.Note that,

$\frac{\partial J(\xi)}{\partial \xi_i} = \frac{\partial \Phi(x(T|\xi))}{\partial x} \frac{\partial x((T|\xi))}{\partial \xi_i}, \ i = 1,2, \dots, N$ (3.2.7)

We need to able to calculate, $\frac{\partial x((T|\xi))}{\partial \xi_1}, \frac{\partial x((T|\xi))}{\partial \xi_2}, \dots, \frac{\partial x((T|\xi))}{\partial \xi_N}$ (3.2.8)

Theorem:

Let $y^{(i)}(t), i = 1,2, \dots, N-1$,satisfy the following delay differential equations:

$$\frac{d\, y^{(i)}(t)}{dt} = \frac{\partial}{\partial\, y^{(i)}} f_{i+1}\left(t, y^i(t), \tilde{y}^{(i)}(t)\right) y^i(t) + \frac{\partial}{\partial \tilde{y}^{(i)}} f_{i+1}\left(t, y^i(t), \tilde{y}^{(i)}(t)\right) \tilde{y}^{(i)}(t),$$

$$\frac{d\, y^{(i)}(t)}{dt} = \frac{\partial}{\partial\, y^{(i)}} f_{i+2}\left(t, y^i(t), \tilde{y}^{(i)}(t)\right) y^i(t) + \frac{\partial}{\partial \tilde{y}^{(i)}} f_{i+2}\left(t, y^i(t), \tilde{y}^{(i)}(t)\right) \tilde{y}^{(i)}(t)$$

(3.2.9)

$\frac{d\, y^{(i)}(t)}{dt} = \frac{\partial}{\partial\, y^{(i)}} f_N\left(t, y^i(t), \tilde{y}^{(i)}(t)\right) y^i(t) + \frac{\partial}{\partial \tilde{y}^{(i)}} f_N\left(t, y^i(t), \tilde{y}^{(i)}(t)\right) \tilde{y}^{(i)}(t)$ with ,

$y^{(i)}(t)|_{t=\tau_i} = f_i\left(t, y^i(t), \tilde{y}^{(i)}(t)\right)|_{t=\tau_i,}$ (3.2.10a)

$y^{(i)}(t-h)| = 0,$ (3.2.10b)

where

$\tilde{y}^{(i)}(t) = y^{(i)}(t-h)$. Then

$\frac{\partial x((T|\xi))}{\partial \xi_1} = y^{(1)}(T), \frac{\partial x((T|\xi))}{\partial \xi_2} = y^{(2)}(T), \dots, \frac{\partial x((T|\xi))}{\partial \xi_{N-1}} = y^{(N-1)}(T)$.

Furthermore $\frac{\partial x((T|\xi))}{\partial \xi_{N-1}} = f_N\left(T, x(T, \xi_{N-1}, \xi_{N-2}, \dots, \xi_1), x(T-h, \xi_{N-1}, \xi_{N-2}, \dots, \xi_1)\right),$

(3.2.11)

Where $x(t, \xi_{N-1}, \dots, \xi_1)$ is solution of system (3.5) corresponding to duration vector ξ.

Proof: Note that $f_i(t, x(t), x(t-h)))$, $i = 1,2 \dots, N$, are continuosly differentiable with respect to their arguments. Thus,by taking the partial differentiation of both sides of(3.5a) with respect to ξ_i, obtain

$$\frac{\partial^2}{\partial \xi_i \partial t}(t; \xi_{i-1}, \dots, \xi_1) = \frac{\partial}{\partial x} f_i(t, x(t; \xi_{i-1}, \dots, \xi_1), \tilde{x}(t; \xi_{i-1}, \dots, \xi_1)) \frac{\partial x}{\partial \xi_i}(t; \xi_{i-1}, \dots, \xi_1)$$

$$+ \frac{\partial}{\partial \tilde{x}} f_i(t, x(t; \xi_{i-1}, \dots, \xi_1), \tilde{x}(t; \xi_{i-1}, \dots, \xi_1)) \frac{\partial \tilde{x}}{\partial \xi_i}(t; \xi_{i-1}, \dots, \xi_1),$$ with

$\tilde{x}(t) = x(t-h)$,since $x(t)$ is dependent on those ξ_j such that $\sum_{j=1}^{i} \xi_j \leq t$, it follows that

$$\frac{\partial x}{\partial \xi_i}(t; \xi_j, \xi_{j-1}, \dots, \xi_1) = 0, \quad if\ t \leq \sum_{k=1}^{j} \xi_k,\ i > k.$$

Let $y^{(i)}(t;\ \xi_k, \xi_{k-1}, \dots, \xi_1) = \frac{\partial x}{\partial \xi_i}(t;\ \xi_k, \xi_{k-1}, \dots, \xi_1), k = 1,2, \dots, N-1.$

New Results and Open Problems for Optimal Control Problem

3.3 Switched Systems

Definition: A switched system is a tuple $s = (F, D)$ where F=$\{f_i: R^n \times R^m \to R^n, i\epsilon I\}$ with f_i is the vector field for the ith subsystem $\dot{x} = f_i(x, u). I = \{1,2, \dots, M\}$ is the set of indices of subsystems.

D=(I, E) is a simple finite state machine which can viewed as directed graph. I serves as the set of discrete states indexing the subsystems. $E \subseteq I \times I - \{(i, i) | i \in I\}$ is a collection of events. If an event $e = (i, j)$ takes place, the switched system will switvh from subsystem i to j.

Aswitched system is a collection of subsystems which are related by a switching logic restricted by D.

The continuous state x and contnuous input u satisfy $x \in R^n$ and $u \in R^m$. Then switched system can be described as

$$\dot{x} = f_{i(t)}(x(t), u(t)) \tag{3.3.1}$$

$$i(t) = \psi(x(t), i(t^-), t) \tag{3.3.2}$$

Where $\psi: R^n \times I \times R \to I$ determines the active subsytem at time t.

Note: If $f_i(x, u)$=$f_i(x)$,$\forall i \in I$,then switched system is said to be autonomous.

Definition: For swiched system S, a switching sequence $\sigma\ in[t_0, t_f]$ is defined as

$$\sigma = \left((t_0, e_0), (t_1, e_1), \dots, (t_K, e_K)\right), \tag{3.3.3}$$

With $0 \le K < \infty, t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$ and $e_0 = i_0 \in I, e_k = (i_{k-1}, i_k) \in E$

for $k = 1,2, .., K$ and i_k is active in $[t_k, t_{k+1})$ if $t_k < t_{k+1}$ ($[t_{K-1}, t_K]\ if\ k = K - 1$), and i_k is switched trough at instant $t_k\ if\ t_k = t_{k+1}$. For switched system we consider nonZeno sequences which switch at most finite number of times in$[t_0, t_f]$. For a switched system to be well behaved ,we generally exclude undesirable Zeno phenomenon.

Note: In thesis we assume that a switching is external in the sense that it is forced by a designer.

An optimal control problem

Problem 1 For a switched system $S = (D, F)$,find a switching sequence $\sigma\epsilon\Sigma_{[t_0,t_f]}$ and an input $u\epsilon U$ such that cost functional $J = \psi\left(x(t_f)\right) + \int_{t_0}^{t_f} L\big(x(t), u(t)\big)dt$ (3.3.4)

is minimized,where t_0, t_f $and\ x(t_0) = x_0$ are given $\psi: R^n \to R, L: R^n \times R^m \to R$.

This problem is fixed final time,free final state problem.

Two stage optimization

Problem 1 requires the solution of an optimal control input (σ^*, u^*) such that

$$J(\sigma^*, u^*) = min_{\sigma\epsilon\Sigma_{[t_0,t_f]},u\epsilon U} J(\sigma, u) \quad (3.3.5)$$

Note that, for any given switching sequence σ ,Problem 1 reduces to a conventional optimal control problem for which we only need to find an optimal continuous input u as to minimize

$J_\sigma(u) = J(\sigma, u)$. The following lemma provides a way to formulate (5) into a two stage optimization problem.

Lemma: For Problem 1, if

(1) an optimal solution (σ^*, u^*) exists and
(2) for any given switching sequence σ ,there exists a corresponding $u^* = u^*(\sigma)$ such that $J_\sigma(u)$ is minimized,then following euations hold

$$\min_{\sigma\epsilon\Sigma_{[t_0,t_f]},u\epsilon U} J(\sigma, u) = min_{\sigma\epsilon\Sigma_{[t_0,t_f]},}\ min_{u\epsilon U} J(\sigma, u) \quad (3.3.6\ )$$

Two stage optimization method

Stage 1 Fixing σ ,solve the iner minimization method.

Stage 2 Regarding the optimal cost for each σ as a function $J_1 = J_1(\sigma)$, minimize J_1 with respect to $\sigma\epsilon\Sigma_{[t_0,t_f]}$.

This method is difficult to handle.From here the above method is using.

Algorithm

1 . Fix the total number of switching s to be $K$and the order of active subsystems,let the minimum value of J with respect to u be a function of the switching instants,i.e, $J_1 = J_1(t_{1,}t_2, \dots, t_K)$ for $K \geq 0$ and then find J_1.

2. (a) Minimize $J_1$with respect to $t_{1,}t_2, \dots, t_K$.

(b) Vary the order of active subsystems to find an optimal solution under K switchings.

(c) Vary the number of switchings K to find an optimal solution for problem 1.

Note : This algorithm has high computational costs.In practice we usually find suboptimal solutions with fixed number of switchings by using steps 1,2(a),2(b).

The variational approach to optimal control problems

We derive necessary conditions for optimal control assuming that the admissible controls are not bounded.

Necessary conditions for optimal control

The problem is to find an admissible control u^* which causes the system

$$\dot{x}(t) = a(x(t), u(t), t) \tag{3.3.7}$$

To follow an admissible trajectory x^* minimizes the performance measure

$$J(u) = h(x(t_f)) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{3.3.8}$$

admissible state and control regions are not bounded also $x(t_0) = x_0$, t_0 are specified.

Usually x is $n \times 1$ state vector and u is $m \times 1$ vector of control inputs. The m control inputs are independent functions.

Assume that h is differentiable function , then,

$$h(x(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}[h(x(t), t)]\, dt + h(x(t_0), t_0) \tag{3.3.9}$$

so performance measure can be expressed as

$$J(u) = \int_{t_0}^{t_f} \left\{ g(x(t), u(t), t) + \frac{d}{dt}[h(x(t), t)] \right\} dt + h(x(t_0), t_0) \tag{3.3.10}$$

$x(t_0)$ and t_0 are fixed and minimization does not affect the $h(x(t_0), t_0)$, so we think about only

$$J(u) = \int_{t_0}^{t_f} \left\{ g(x(t), u(t), t) + \frac{d}{dt}[h(x(t), t)] \right\} dt \tag{3.3.11}$$

Using chain rule of differentiation,

$$J(u) = \int_{t_0}^{t_f} \left\{ g(x(t), u(t), t) + \left[\frac{\partial h}{\partial x}(x(t), t)\right]^T \dot{x}(t) + \frac{\partial h}{\partial t}(x(t), t) \right\} dt \tag{3.3.12}$$

From differential equation constraints,we form augmented functional

$$J_a(u) = \int_{t_0}^{t_f} \left\{ \begin{matrix} g(x(t), u(t), t) + \left[\frac{\partial h}{\partial x}(x(t), t)\right]^T \dot{x}(t) + \frac{\partial h}{\partial t}(x(t), t) + \\ p^T(t)\left[a(x(t), u(t), t) - \dot{x(t)}\right] \end{matrix} \right\} dt \qquad (3.3.13)$$

$p_1(t), \dots, p_n(t)$ is lagrange multipliers.

$$g_a(x(t), \dot{x}(t), u(t), p(t), t) \triangleq g(x(t), u(t), t) + p^T(t)\left[a(x(t), u(t), t) - \dot{x(t)}\right]$$

$$+ \left[\frac{\partial h}{\partial x}(x(t), t)\right]^T \dot{x}(t) + \frac{\partial h}{\partial t}(x(t), t)$$

so that

$$J_a(u) = \int_{t_0}^{t_f} \{g_a(x(t), \dot{x}(t), u(t), p(t), t)\}\, dt \qquad (3.3.14)$$

Assume that $t = t_f$ can be specified or free. To determine J_a we define $\delta x, \delta\dot{x}, \delta u, \delta p, \delta t_f$ so

$$\delta J_a(u) = 0 = \left[\frac{\partial g_a}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f\right)\right]^T \delta x_f$$

$$+ \left[g_a(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f - \left[\frac{\partial g_a}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f\right)\right]^T \dot{x}^*(t_f)\right] \delta t_f$$

$$+ \int_{t_0}^{t_f} \left\{ \begin{matrix} \left[\left[\frac{\partial g_a}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f\right)\right]^T - \right. \\ \left. \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f\right)\right]^T\right] \delta x(t) + \\ \left[\frac{\partial g_a}{\partial u}\left(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f\right)\right]^T \delta u(t) + \\ \left[\frac{\partial g_a}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f\right)\right]^T \delta p(t) \end{matrix} \right\} \qquad (3.3.15)$$

Notice that $\dot{u}(t)$ and $\dot{p}(t)$ do not appear.

$$\frac{\partial}{\partial x}\left[\left[\frac{\partial h}{\partial x}(x^*(t), t)\right]^T \dot{x}^*(t) + \frac{\partial h}{\partial t}(x^*(t), t)\right] - \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{x}}\left[\left[\frac{\partial h}{\partial x}(x^*(t), t)\right]^T \dot{x}^*(t)\right]\right\} \qquad (3.3.16)$$

Writing out the indicated partial derivatives gives us

$$\left[\frac{\partial^2 h}{\partial x^2}(x^*(t), t)\right] \dot{x}^*(t) + \left[\frac{\partial^2 h}{\partial t \partial x}(x^*(t), t)\right] - \frac{d}{dt}\left[\frac{\partial h}{\partial x}(x^*(t), t)\right] \qquad (3.3.17)$$

With applying chain rule

$$\left[\frac{\partial^2 h}{\partial x^2}(x^*(t), t)\right] \dot{x}^*(t) + \left[\frac{\partial^2 h}{\partial t \partial x}(x^*(t), t)\right] - \left[\frac{\partial^2 h}{\partial x^2}(x^*(t), t)\right] \dot{x}^*(t) - \left[\frac{\partial^2 h}{\partial t \partial x}(x^*(t), t)\right]$$

ıf assume that second order partial derivatives are continus,the order of differentiation can be interchanged ,and these terms add to zero.In the integral term ,

$$\int_{t_0}^{t_f}\left\{\left[\left[\frac{\partial g}{\partial x}(x^*(t),u^*(t),t)\right]^T+p^{*T}(t)\left[\frac{\partial a}{\partial x}(x^*(t),u^*(t),t)\right]-\frac{d}{dt}[-p^{*T}(t)]\right]\delta x(t)+\left[\left[\frac{\partial g}{\partial u}(x^*(t),u^*(t),t)\right]^T+p^{*T}(t)\left[\frac{\partial a}{\partial u}(x^*(t),u^*(t),t)\right]\right]\delta u(t)+[[a(x^*(t),u^*(t)-\dot{x}^*(t)]^T]\delta p(t)\right\}dt. \quad (3.3.18)$$

This integral must vanish on extremal regardless of the boundary conditions. First observe that

$$\dot{x}(t)=a(x^*(t),u^*(t),t)) \quad (3.3.19a)$$

must be satisfied by an extremal so that coefficient of $\delta p(t)$ is zero. Lagrange multipliers are arbitrary so $\delta x(t)$is zero that is

$$\dot{p}^*(t)=-p^{*T}(t)\left[\frac{\partial a}{\partial x}(x^*(t),u^*(t),t)\right]^T-\frac{\partial g}{\partial x}(x^*(t),u^*(t),t) \quad (3.3.19b)$$

. $\delta u(t)$ is independent so its coefficient must be zero

$$0=\frac{\partial g}{\partial u}(x^*(t),u^*(t),t)+\left[\frac{\partial a}{\partial u}(x^*(t),u^*(t),t)\right]^T p^*(t) \quad (3.19c)$$

The variation must be zero so

$$\left[\frac{\partial h}{\partial x}x^*(t_f),t_f)-p^*(t_f)\right]^T\delta x_f+\left[g(x^*(t_f),u^*(t_f),t_f)+\frac{\partial h}{\partial t}x^*(t_f),t_f)+p^{*T}(t_f)[a(x^*(t_f),u^*(t_f),t_f)]\right]\delta t_f=0 \quad (3.3.20)$$

. It is important that these necessary conditions consist of a set of $2n$,first order differential equations . The solution of the state and costate equations will contain $2n$ constants of integration. To evaluate these constants using n equations $x^*(t_0)=x_0$ and additional set of n or ($n+1$) relationships depending on whether or not t_f is specified from equation (15). In the following find it convenient to use the function H called Hamiltonian ,defined as

$$H(x(t), u(t), p(t), t) \triangleq g(x(t), u(t), t) + p^T(t)[a(x(t), u(t), t)]. \qquad (3.3.21)$$

We can write necessary conditions:

$$\left.\begin{aligned} \dot{x}^*(t) &= \frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) \\ \dot{p}^*(t) &= -\frac{\partial H}{\partial x}(x^*(t), u^*(t), p^*(t), t) \\ 0 &= \frac{\partial H}{\partial u}(x^*(t), u^*(t), p^*(t), t) \end{aligned}\right\} \quad \text{for all } t \epsilon [t_0, t_f]. \qquad (3.3.22)$$

And

$$\left[\frac{\partial h}{\partial x}\left(x^*\left(t_f\right), t_f\right) - p^*\left(t_f\right)\right]^T \delta x_f +$$

$$\left[H(x^*\left(t_f\right), u^*\left(t_f\right), p^*\left(t_f\right), t_f + \frac{\partial h}{\partial t}\left(x^*\left(t_f\right), t_f\right)\right] \delta t_f = 0$$

CONCLUSION

In thesis, reults for optimal control for hybrid system are reported. We studied optimal control problems for switched systems in which prespecified sequence of active subsystem is given. Idea of two stage optimization , proposed method to obtain accurate values of derivatives that is necessary for stage (b). This method is trancribes an optimal control problem into an equivalent parametrized by the switching instants and derives the derivatives based on solution of a two boundary value formed by state, costate, stationary equations, the boundary and and continuity conditions.

Then, we considered class of optimal control problems governed by switched systems with time delay. We parametrized switching instants and derived the required gradient of cost function, which some delay differential equations are required to be solved forward in time.

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