T.C

(MASTER THESIS)

YAŞAR UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

# *T*-NONCOSINGULAR ABELIAN GROUPS

Surajo SULAIMAN

Thesis Advisor: Prof. Dr. Rafail ALIZADE

MATHEMATICS DEPARTMENT

Barnova-IZMIR

June-2014

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# APPROVAL PAGE

This study, title "*T*-noncosingular Abelian Groups" and presented as Master Thesis by Surajo SULAIMAN has been evaluated in compliance with the provisions of Yaşar University Graduate Education and Training Regulation and Yaşar University Institute of Science Education and Training Direction. The jury members below have decided for the defence of this thesis and it has been declared by consensus/majority of the votes that the candidate has succeeded in his thesis defence examination dated 13$^{th}$June, 2014

Jury Members

Prof. Dr. Rafail ALIZADE ..........................................
Yaşar University
(Thesis supervisor) Signature and Date

Prof. Dr. Mehmet Terziler ..........................................
Yaşar University
(Member) Signature and Date

Assoc. Prof. Dr. Engin Büyükaşir ..........................................
Izmir Institute of Technology
(Member) Signature and Date

# ABSTRACT

# *T*-NONCOSINGULAR ABELIAN GROUPS

SULAIMAN, Surajo
MSc, Department of Mathematics
Supervisor: Prof. Dr. Rafail ALIZADE
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In this Thesis, we study $T$-noncosingular abelian groups, that is abelian groups whose nonzero endomorphisms are not small. We show that injective (divisible) and projective (free) groups are $T$-noncosingular. We prove that $T$-noncosingular torsion groups are exactly the direct sum of a semisimple group C and a divisible group $D$ which does not have simple subgroups isomorphic to a subgroup of C. We also give some condition for torsion-free groups to be $T$-noncosingular.

**Keywords: -** Abelian group, Torsion group, Torsion-free group, $T$-noncosingular, small homomorphims, small subgroup, simple group and semi-simple group.

# ÖZET

## *T*- EŞTEKIL OLMAYAN DEĞİŞMELİ GRUPLAR

SULAIMAN, Surajo
Yüksek Lisans, Matematik Bölümü
Tez Danişmani: Prof. Dr. Rafail Alizade
Haziran 2014, 46 safya.

Bu tezde $T$ -eştekil olmayan, yani sıfırdan farklı endomorfizmaları küçük olmayan değişmeli grupları inceliyoruz. İnjektif (bölünebilir) ve projektif (serbest) grupların $T$-eştekil olmadığını gösteriyoruz. Buralmalı $T$-eştekil olmayan grupların tam olarak, $C$ ile $D$ isomorf basit alt grup içermeyecek şekilde yarıbasit $C$ ve bölünebilir $D$ gruplarının dik toplamı olduğunu kanıtlıyoruz. Ayrıca burulmasız grupların da $T$ -eştekil olmaması için bir yeterli koşul veriyoruz.

Anahtar Sözcükler: Değişmeli grup, burulma grubu, burulmasız grup,$T$ -eştekil olmayan grup, küçük alt grup, küçük homomorfizma, basit grup ve yarıbasit grup.

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Surajo SULAIMAN

Izmir, 2014

# TEXT OF OATH

I declare and honestly confirm that my study, title “T-noncosingular Abelian Groups” and presented as a Master’s Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the references, and that I have benefited from these sources by means of making references.

$13-06-2014$ ........................................................

Student Name and Signature

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| | |
|---|---|
| $\mathbb{Z}$ | The group of integers |
| $\mathbb{Q}$ | The group of rational numbers |
| $\mathbb{Z}_{p^\infty}$ | Quasi-cyclic group ( primary part of torsion group $\mathbb{Q}/_{\mathbb{Z}}$ ) |
| G$[n]$ | The sets of all $g \in G$ with $ng = 0$ |
| $n$G | The sets of all $ng$ with $g \in G$. |
| $d$G | Maximal divisible subgroup of an abelian group |
| $t$G | Torsion subgroup of an abelian group |
| $\prod A_i$ | Direct product of groups |
| $\sum A_i$ | Sum of abelian groups |
| $A \oplus B$ | Direct sum of abelian groups |
| Hom(G, N) | All homomorphism from G to N |
| $Kerf$ | Kernel of the map $f$ |
| $Imf$ | Image of the map $f$ |
| Rad G | Radical of a group G |
| $\mathbb{Z}_n$ | Integers modulo n |
| $\cap$ | Intersection of sets |
| $\ll , \trianglelefteq$ | Small (superfluous), Essential |
| $\leq$ | Subgroup |
| $\cong$ | Isomorphism (Isomophic) |
| $A/_B$ | Quotient group |

# CHAPTER ONE

## INTRODUCTION

Abelian groups play an important role in a modern approach to Abstract Algebra. Really it is used to define certain concept like Module and Vector spaces. The notion of *T*- noncosingular Module which will be explain later in Chapter five was introduced and studied by of D.K Tutuncu and R Tribak in 2009 in the Paper "On *T*- noncosingular Module". In this thesis work we will look at the same notion but in our case will be a further restriction.

In 2009 Derya Keskin Tutuncu and Rashid Tribak introduced and studied the concept of *T*-noncosingular Modules[13] and their work was due to the concept (which is a dual) of *K*-nonsigular modules and application presented by S.T Rizvi and C.S. Roman[10] in 2007. The actual concept of *K*-noncosingular was introduced by Rizvi and Roman in the paper "Bear and Bear modules" in 2004 and this paper was from the Doctoral Dissertation of Roman C. S. (2004) in Ohio state university. Also in 2013 Rashid Tribak presented some result on *T*-noncosingular Modules [14]. In 2010 Ozan Gunyuz also in his MSc thesis presented and studied some further notion which they defined as Strongly *T*-noncosingular Modules.

In the view of the above we present the notion of *T*-noncosingular Abelian groups and since an abelian group is a $\mathbb{Z}$-module, we shall use most of the definitions and properties of modules satisfying $\mathbb{Z}-$modules for the Abelian groups.

Does *T*-noncosingular Abelian group exist? Can we characterize it? These and some other questions will be attempted in this thesis. This notion required the knowledge of different abelian groups and certain subgroups such as small and essential subgroup in addition to pure and basic subgroup, which will be presented later in this work.

A group $G$ is said to be *T*-noncosingular if $imf$ is not small in $G$ for every nonzero endomorphism $f$ of $G$.

We start chapter two with the basic ideas on groups, subgroups, homomorphism, isomorphism, direct sums and direct products and rounded the chapter with injective and projective groups. Chapter three focuses on more topics on group theory such as torsion and torsion-free group, we also touch divisible groups, $p$-groups, pure and basic subgroups and other subject related to our topic of this thesis. Chapter four focuses on small subgroup, essential subgroup, semisimple group and rounded with radical of an abelian group, chapter five will be the most important part of this thesis where our original work will be presented and finally this work will be rounded off with the conclusions on our result from chapter five, which is the chapter six of this research work.

Starting from chapter two, examples, theorems, corollaries, lemmas, propositions are given to carry the reader along especially chapter five of this work where many result will be used to generalized the concept of the research work along the line of two main notions of abelian group, that is, torsion and torsion-free. If the reader has some knowledge of the abelian group, he can read chapter three and four briefly before going to chapter five.

# CHAPTER TWO

## 2.1 Motivation

This chapter gives short introduction of the abelian group theory for the reader to fully understand the thesis, but for details on groups and abstract algebra in general, the reader can see [3],[4],[5],[7] and for advanced group theory the reader can see [2],[6],[9], if the reader has a good understanding of the group theory, then he can go to chapter three and read it briefly before going to chapter four, while chapter five focuses on the most important parts of this thesis.

**Definition 2.01** A group $< G,*>$ is a set $G$ closed under the binary operation * such that the following axioms are satisfied:

$M_1$: For $a, b, c \in G$ $(a * b) * c = a * (b * c)$. (Associativity of $*$)

$M_2$: There is an element $e$ in $G$ such that for all $a \in G, e * a = a * e$ , (identity element $e$ for $*$).

$M_3$: Corresponding to each $a \in G$, there is an element $a'$ in $G$ such that $a' * a = a * a' = e$, (inverse $a'$ of $a$).

Note that for the purpose of this research thesis we will concentrate on Abelian groups, as such the operation * will be replace with + and the identity element will simply be 0 while the inverse of any element $(a)$ will be $(-a)$. We write $na = a + a + a \ldots + a$,(n-times) with $a \in G, n \in \mathbb{Z}^+$, if for $n \in \mathbb{Z}^+$ and $a \neq 0, na = 0$, then the order of that element $a$ is $n$ will be denoted as $o(a) = n$. By a group we will mean an abelian group.

The following are some examples of Abelian groups ($\mathbb{R}$, +), ($\mathbb{Q}$, +) and ($\mathbb{Z}$, +), but there are also non-abelian groups which will not be our area of discussion for the purpose of this research work.

Sub-structures (subsets) of a bigger structure in most cases form what we call subgroup of a giving group provided it preserves the structure of the bigger group under the same operation, for example $\mathbb{Z}, \mathbb{Q}$ are subset $\mathbb{R}$, thus ($\mathbb{Q}$, +) and ($\mathbb{Z}$, +) are subgroups of ($\mathbb{R}$, +), going by this rule one can see that ($\mathbb{Z}$, ·) is not a subgroup of ($\mathbb{R}$, +) since the operations differ, for example, for every $a, b, \in \mathbb{Z}$, $a \cdot b$ may not be $a + b$ in general. With this we can now give the definition of a subgroup as follows.

**Definition 2.02** If a subset $H$ of a group $G$ is closed under the operation of $G$ and the subset $H$ with that operation form a group, then $H$ is called a subgroup of a group $G$.

The reader may note that if $G$ is abelian so is $H$ as a subgroup of $G$, and will be denoted as $\leq$, we write $H \leq G$ if $H$ is a subgroup of a group $G$ and $H < G$ if $H \leq G$, but $H \neq G$.

If $H < G$ then $H$ is a proper subgroup, otherwise $H$ is just a subgroup, but if $H = G$ then $H$ will be called an improper subgroup of a group $G$, and lastly $\{0\}$ is a trivial subgroup of any group $G$. Finally we will give the generalization for any subset to be a subgroup.

**Theorem 2.1.1**[Subgroup Test, (15, 1.2.10)] Let $H$ be a subset of $G$. Then $H \leq G$ if and only if $0 \in H$ and for $x, y \in H$ then $x - y \in H$.

Any subset satisfying above criterion will be called a subgroup of a given group.

**Definition 2.03**(Cyclic Subgroup) Let $h$ be an element of $G$, then a set $H$=$\{nh$ / n$\in \mathbb{Z}\}$ is called a cyclic subgroup generated by an element h and is denoted by $H$=$< h >$, it is the smallest subgroup which contains $H$.

## 2.2 Homomorphism and Isomorphism

The concept of homomorphism is no doubt one of the most important notions of the group theory. It provides us with much information concerning the structure of the other group.

For an isomorphism this gives more information, because the map must be onto and one-to-one, so they may be structurally the same with the first group.

**Definition 2.04** Let $H$ be a subgroup of a group $G$. The subset $g + H = \{g + h \mid h \in H\}$ of $G$ is the left coset of $H$ generated $g$. Thus $H + g$ will be called the right coset, but since we are concern with only abelian groups, the left and the right coset coincide (Every subgroup is a normal subgroup)

Example 2.1 Describe all cosets of the subgroup $4\mathbb{Z}$ $of$ $\mathbb{Z}$

$$4\mathbb{Z} = \{- - - - - - - - - - -, -8, -4, 0, 4, 8, - - - - - - - - -\}$$

$$1+4\mathbb{Z} = \{- - - - - - - - - - -, -7, -3, 1, 5, 9, - - - - - - - - - - -\}$$

$$2+4\mathbb{Z} = \{- - - - - - - - - -, -6, -2, 2, 6, 10, - - - - - - - -\}$$

$$3+4\mathbb{Z} = \{- - - - - - - - - - -, -5, -1, 3, 7, 11, - - - - - - - - -\}$$

Note that, cosets partition the group into many disjoint subsets of $G$which may or may not be a subgroup of $G$.

**Theorem 2.2.1** [15, 1.6.1] Let $H$ be a subgroup of an abelian group $G$. Then the set $G/_H$ together with the operation $(a+H)+(b+H)=(a+b)+H$ form a group called quotient or factor group of the group $G$ mod $H$.

**Definition 2.05** Let $H$ be a subgroup of the group $G$, then the coset of $H$ denoted by $G/_\mathrm{H}$ is called a factor group or a quotient group of $G$.

**Example 2.2** gives the details of cosets of the subgroup $4\mathbb{Z}$ of $\mathbb{Z}$, therefore $\{4\mathbb{Z}, 1+4\mathbb{Z}, 2+4\mathbb{Z}, 3+4\mathbb{Z}\}$ forms factor group of $\mathbb{Z}$. and defined as $\mathbb{Z}/_{4\mathbb{Z}}$. we shall see later in example 2.2 that $\mathbb{Z}/_{4\mathbb{Z}} \cong \mathbb{Z}_4$, from isomorphism theorems.

The reader may note that, the order of the factor group $G/_H$ is O $(G:H)$ and this may be due to the famous Lagrange theorem.

**Definition 2.06** A function (map) $f$ of a group $G$ into a group $H$ is a homomorphism if it satisfies the condition that $f(a+b)=f(a)+f(b)$ for all $a,b \in G$.

The reader may note that there is always the trivial homomorphism defined by $f(g)=0$ for all $g \in G$

**Definition 2.07** The homomorphism $f$ form $G$ into $H$ is called a monomorphism if $f$ is one-to-one and $f$ $is$ an epimorphism, if $f$ is onto mapping, while $f$ is an endomorphism, if $f$ maps form $G$ to $G$ itself.

**Definition 2.08** A monomorphism that is also an epimorphism is called an isomorphism. An isomorphism from $G$ to $G$ itself is called an automorphism.

PROPERTIES OF HOMOMORPHISMS

Following [5, 13.12] it is clear that the following are the properties of a homomorphism.

**Theorem 2.2.2** let $f$ be a homomorphism from a group $G$ into a group $H$.

- If 0 is the identity element in $G$, then $f(0)$ is the identity element in $H$
- İf $g \in G, then\, f(g^{-1}) = f(g)^{-1}$
- If $K$ is a subgroup of $G$ then $f[H]$ is a subgroup of $H$
- If $K'$ is a subgroup of $H$, then $f^{-1}[K']$ is a subgroup of $G$.

**Definition 2.09** If $f\colon G \longrightarrow H$ is a homomorphism then $f[G]$ is called the image of $f$ and is denoted by $Imf$.

**Definition 2.10** If $f\colon\ G \longrightarrow H$ is a homomorphism then $f^{-1}(0) = \{g \in G\ |f(g) = 0\}$ is called the $k$ernal of $G$ and is denoted by K$er\ f$.

**Corollary 2.2.3** [1] A homomorphism $f$ from $G$ into $H$ is one-to-one if and only if K$erf = \{0\}$.

Proof: suppose that $f\ is\ 1-1$, let$g \in kerf$, then $f(g) = 0 = f(0)$

This means that $f(g) = f(0)$ , since $f\ is\ 1-1$, we have $g = 0$.

Suppose that $kerf = 0$, let $f(x) = f(y)$, then $f(x) - f(y) = 0$

$then\ f(x-y) = 0$. This means that $x - y \in kerf$ then $x - y = 0$ which gives $x = y$

**Definition 2.11** A homomorphism $f\colon G \longrightarrow H$ is onto if $Imf = H$

**Theorem 2.2.4** [5, 14.9] Let $H$ be a subgroup of a group $G$. Then a function $f\colon G \longrightarrow {}^{G}/_{H}$ defined by $f(x) = x + H$ is a homomorphism with $kerf = H$.

Now we are ready to introduce the reader to another important concept of the group theory which we often used as a tool in our routine research.

**Theorem 2.2.5** [Isomorphism theorem, (15, 1.6.3)]:- Let the function $f$ be a homomorphism from $G$ onto$H$. Then $G/_{kerf} \cong H$.

**Example 2.3** Consider the function $f: \mathbb{Z} \longrightarrow \mathbb{Z}_n$ defined by $f(m) = r$ where $r$ is the remainder when dividing $m$ by $r$. Then we can immediately see that $kerf = n\mathbb{Z}$ and by above theorem, we can write $\mathbb{Z}/_{n\mathbb{Z}} \cong \mathbb{Z}_n$ in general and remember that by putting $n = 4$, in the result we have $\mathbb{Z}/_{4\mathbb{Z}} \cong \mathbb{Z}_4$ , from now on we will consider $\mathbb{Z}/_{n\mathbb{Z}}$ and $\mathbb{Z}_n$ to be algebraically the same.

**Theorem 2.2.6** [Second isomorphism theorem, (11, 1.6.6)]:-Let $H$ and $N$ be subgroups of $G$, then $H + N/_N \cong H/_{H \cap N}$ .

**Definition 2.12** Let $G$ be a group and $H$ be a subgroup, then $f: G \longrightarrow G/_H$ defined as $f(g) = g + H$ is the natural or canonical epimorphism.

**Theorem 2.2.7** [Third isomorphism theorem, (15, 1.6.6)] Let $N$ and $H$ be subgroups of $G$ with N ⊆ H, then $G/_H \cong G/_N/H/_N$ .

**Example 2.3** Take $f: \mathbb{Z}/_{6\mathbb{Z}} \longrightarrow \mathbb{Z}/_{2\mathbb{Z}}$ define by $f(a + 6\mathbb{Z}) = a + 2\mathbb{Z}$, we can see that, the $kerf = 2\mathbb{Z}/_{6\mathbb{Z}}$ and by first isomorphism theorem, we will have

$$\mathbb{Z}/_{2\mathbb{Z}} \cong \left. \mathbb{Z}/_{6\mathbb{Z}} \middle/ 2\mathbb{Z}/_{6\mathbb{Z}} \right.$$ (from the third isomorphism theorem),

## 2.3 Direct Sum and Direct Product

Like homomorphism the concept of direct sum plays an important role in the theory of an Abelian group. Sometimes the structure of the group can easily be seen in case of finite group, but in most cases we use decomposition to study structure of the group and even use the result to construct some new groups. There are mainly two types of direct sum (internal and external direct sum).

**Definition 2.13** Let $H$ and $N$ be subgroups of a group $G$, if $H + N = G$, and $H \cap N = 0$, then $G$ is called the (Internal) direct sum of $H$ and $N$ and is denoted as $G = H \oplus N$.

From above definition we can further generalise the concept by taking family of subgroup of G (finite or infinite) as follows.

**Definition 2.14** Let $(H_i)_{i \in I}$ be a family of subgroups of $G$, if $\Sigma H_i = G$ and $H_i \cap \sum_{i \neq j} H_j = 0$, then $G$ is said to be a direct sums of $H_i$ and denoted $as\ G = \bigoplus_{i \in I} H_i$.

**Definition 2.15** A subgroup $H$ of $G$ is called a direct summand, if there is a subgroup $N \leq G$ such that $G = H \oplus N$. Also $N$ is called complentary direct summand or complement of $H$ in $G$.

**Definition 2.16** For an element $g$ of a group $G$, the order of that element is the smallest positive integer n such that $ng = 0$ and is denoted by $o(g) = n$.

Note that, if for any element $g$ there is no positive integer n such that $ng$ = 0 then $g$ is said to have an infinite order.

Following [9, page 38, (Fuchs, 1970)] The following are the properties of an internal direct sum.

If $G = B \oplus C$, then $C \cong {}^{G}/_{B}$. ( Thus the complement of $B$ in $G$ is unique up to isomorphism )

1) If $G = B \oplus C$, and if $A$ is a subgroup of $G$ containing $B$ then $A = B \oplus (A \cap C)$.
2) For $g \in G = B \oplus C$, and if $g = b + c$ $(b \in B, c \in C)$, then o$(g)$ is the least common multiple of the $o(b)$ and $o(c)$.
3) If $G = \bigoplus_i B_i$, and if , for every $i, C_i \leq B_i$, then $\sum C_i = \bigoplus C_i$, this is a proper subgroup of G if $C_i < B_i$ for at least one $i$.
4) If $G = \bigoplus_i B_i$ where each $B_i$ is a direct sum, and $B_i = \bigoplus_j B_{ij}$, then $G = \bigoplus_i \bigoplus_j B_{ij}$.
5) If $G = \bigoplus_i \bigoplus_j B_{ij}$, then G=$\bigoplus_i B_i$ with $B_i = \bigoplus_j B_{ij}$.

**Definition 2.17** Let $(H_i)_{i \in I}$ *and* $\prod_{i \in I} H_i = \{(h_i) \mid h_i \in H_i$, then $\prod_{i \in I} H_i$ is called the direct product of $(H_i)_{i \in I}$.

The reader may verify that the concept of direct sum and direct product coincide if $I$ is finite.

**Definion 2.18** The direct sum (or weak direct sum), denoted by $\sum_{k\in K} A_k$ is the subgroup of $\prod_{k\in K} A_k$ consisting of all those elements $(a_k)$ for which there are only finitely many $k$ with $a_{\mathrm{k}} \neq 0$.

**Definition 2.19** (External Direct sum) Suppose that $G = A \oplus B$ and let $A' = \{(a,0) \mid a \in A\}$, and $B' = \{(0,b) \mid b \in B\}$ and define one-to-one and onto map from $A$ to $A'$ as $f(a) = (a,0)$, as such $A \cong A'$, also using the same pattern we have $B \cong B'$, then $G = A \oplus B \cong A' \oplus B'$, this means that $G = A' \oplus B'$. So the external direct sum of $A$ and $B$ is isomorphic to the internal direct sum of subgroups $A'$ and $B'$ isomorphic to $A$ and $B$, further we will not distinguish the internal and external direct sums.

**Lemma 2.3.1** [11, 10.3] If $G$ is an Abelian group and $A \leq G$, then the following statement are equivalent.

i) A is a direct summand of $G$, that there exists a subgroup $B$ of $G$ with $A \cap B = 0$ and $A + B = G$.

ii) There is a subgroup $B$ of $G$ so that each $g \in G$ has a unique expression $g = a + b$ with $a \in A \; and \; b \in B$.

iii) There exists a homomorphism $\varphi : {}^{G}/_{A} \longrightarrow G$, with $\sigma\varphi = 1_{G/_{A}}$, where $\sigma: G \longrightarrow {}^{G}/_{A}$, (Canonical Map).

iv) There exists a retraction $\pi: G \longrightarrow A, such\ that\ \pi\ is\ a$ homomorphism with $\pi(a) = a$ for all $a \in A$.

**Theorem 2.3.2** [8,(factor theorem)]:- Let $f: A \longrightarrow B$ be a homomorphism and $g: A \longrightarrow C$ be an epimorphism with $kerg \leq kerf$. Then there exist a homomorphism $h: C \longrightarrow B$ with

1) $f = h^\circ g$
2) $Imh = Imf$
3) $h$ is a homomorphism if $kerg \leq kerf$

Motivated by the above theorem we can state the following corollary

**Corollary 2.3.3** Let $f\colon A \longrightarrow B$ be a homomorphism and $\sigma : A \longrightarrow {}^{A}/_{kerf}$ be canonical epimorphism, then there exist a homomorphism $h\colon {}^{A}/_{kerf} \longrightarrow B$ such that $f = h^{\circ}g$ as shows by the diagram below

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
{}^{A}/_{kerf} \arrow[ur, "h"'] &
\end{tikzcd}$$

$fig$ 2.1

<u>Note</u>

If $f$ is a monomorphism then ${}^{A}/_{kerf} \cong A$, so that $g$ will be an endomorphism.

## 2.4 Injective and Projective Abelian Group

To study this topic there is need to learn something about exact sequence of an abelian groups which is presented as follows.

**Definition 2.20** A sequence $G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 -- \xrightarrow{f_k} G_k$ of groups $G_i$ and homomorphism $f_i$ is exact if $Imf_i = Kerf_{i+1}$, for $i = 1,2,...,k-1$

In particular $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if $f$ is monic, while $B \xrightarrow{f} C \longrightarrow 0$ is exact if and only if $f$ is epimorphism, therefore combining the two we have isomorphism and referred to us, as short exact sequence.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

**Definition 2.21** A group $G$ is said to be injective if for every diagram with exact row there exist a homomorphism $h\colon B \longrightarrow G$ making the diagram below to commute.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[rr, "f"] \arrow[d, "g"'] & & B \arrow[dll, "h"] \\
& G & &
\end{tikzcd}$$

$fig$ 2.2

**Proposition 2.4.1** [8, 5.3.4(a)]:- A direct product $\prod_i G_i$ is injective if and only if each $G_i$ is injective.

Above proposition can be further explain using the following diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "g"'] & & B \arrow[dll, "h"] \arrow[ddll, "u"] \\
G_k \arrow[d, shift right, "\alpha_k"'] & & \\
\prod_k G_k \arrow[u, shift right, "p_k"'] & &
\end{tikzcd}$$

$fig$ 2.3

That means if each $G_k$ is injective then, there exist $u: B \longrightarrow \prod_k G_k$ with the condition that $h = p_k \circ u$ and $u \circ f = \alpha_k \circ g$ , $h \circ f = p_k \circ u \circ f = p_k \circ \alpha_k \circ g = 1_{G_k} \circ g = g$, which means above definition make sense.

**Definition 2.22** A group $P$ is projective if for every epimorphism $f: A \longrightarrow B$ and $g: P \longrightarrow B$ there is a homomorphism $h: P \longrightarrow$ A such that $f \circ h = g$

$$\begin{tikzcd}
f\colon A \arrow[rr, "f"] & & B \\
& P \arrow[ul, "h"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$fig$ 2.4

**Proposition 2.4.2** [8, 5.3.4(b)] A direct sum $\oplus G_i$ is projective if only if each $G_i$ is projective.

# CHAPTER THREE

## 3.1 Torsion and Torsion free-group

**Definition 3.01** If $G$ is an Abelian group, then the sets of elements $tG$ = { $g \in G$ | $ng = 0$ for some non-zero integer n} is called a torsion part of the group.

**Definition 3.02** A group $G$ is called a torsion group, if $tG = G$ and torsion-free if $tG = 0$

**Theorem 3.1.1** [11, 10.1] The quotient group $G/_{tG}$ is a torsion-free group.

Proof. If $n(g + tG) = 0$ in $G/_{tG}$ for some $n \neq 0$, then $ng \in tG$, and so there is $m \neq 0$ with $m(ng) = 0$. Since $mn \neq 0$, $g \in tG$, $g + tG = 0$ in $G/_{tG}$, and $G/_{tG}$ is torsion-free.

Following the above theorem one can see that every group is an extension of a torsion group by a torsion free group.

**Definition 3.03** Let $p$ be a prime number, group $G_p$ is called a $p$-group or sometimes a $p$-primary group if $G_p = \{x \in G \mid p^n x = 0$ for some integer $n \in \mathbb{Z}\}$

**Theorem 3.1.2** [11, 10.7] Every torsion group is a direct sum of $p$-primary groups. That is, if $G$ is torsion then $G = \bigoplus_p G_p$ where $p$ is prime.

*Proof.* Since $G$ is torsion, for some integer $n$: we have $nx = 0$ for all $\in G$. Now for each prime divisor $p$ $of$ $n$, define $G_p$ = {$x \in$ G: $p^t x = 0$ for some $t$ }.Now $G_p$ is a subgroup of $G$, for if $p^n x = 0$ and $p^m y = 0$, where $m \leq n$, then $p^n(x - y) = 0$, so $G_p$ is a subgroup. We claim that G = $\sum G_p$ , and we use the following criterion

1) $G_p \cap \sum_{q \neq p} G_q = 0$ where q is prime

2) $G = \sum G_p$

Let $n = p_1^{t_1} \dots\dots\dots\dots\dots.. p_k^{t_k}$, where $p_i$ are the distinct primes and $t_i > 0$ for all $i$. Set $n_i = \frac{n}{p_i^{t_i}}$; and observe that the $gcd(n_1, \dots ,n_k) = 1$. Therefore there are integers $s_i$ with $\sum s_i n_i = 1$, and so $x = \sum s_i n_i x$. But $s_i n_i x \in G_p$ because $p_i^{t_i} s_i n_i x = s_i n x = 0$ . Therefore, $G$ is generated by the family of $G_p$'s. Therefore assume that $x \in G_p \cap \sum_{q \neq p} G_q$ On the one hand, $p^t x = 0$ for some $t > 0$; on the other hand, $x = \sum x_q$, where $q^{t_q} x_q = 0$ for exponents $t_q$. *If* $m = \prod q^{t_q}$, then $m$

and $p^t$ are relatively prime, and there are integers $r$ and $s$ with $1 = rm + sp^t$. Therefore, $x = rmx + sp^t x = 0$, and so $G_p \cap \sum_{q \neq p} G_q = 0$.

**Theorem 3.1.3** [11, 10.8] If $G$ and $H$ are torsion groups, then $G \cong H$ if and only if $G_p \cong H_p$ for all prime $p$.

*Proof.* If $\varphi: G \longrightarrow H$ is a homomorphism, then $\varphi(G_p) \leq H_p$ for all primes $p$.In particular, if $\varphi$ is an isomorphism, then $\varphi(G_p) \leq H_p$ and $\varphi^{-1}(H_p) \leq G_p$ for all $p$. It follows easily that $\varphi \setminus_{G_p}$ is an isomorphism $G_p \longrightarrow H_p$.

Conversely, assume that there is isomorphism $\varphi$: $G_p \longrightarrow H_p$. For all prime $p$. By **Lemma 2.3.1** (ii), each $g \in G$ has a unique expression of the form $g = \sum_p a_p$ where only a finite number of $a_p \neq 0$. Then $\varphi: G \longrightarrow H$ , defined by $\varphi(\sum_{.} a_p)$ $=\sum \varphi_p(a_p)$ is easily se en to be an isomorphism.

QUASI-CYCLIC GROUP

We must state that, this group is an important tool in group theory as many counter examples are given to prove or disprove many claims. We will give some properties of this group and its elements.

Note that $\mathbb{Q}/\mathbb{Z}$ is a torsion group since $n\left(\frac{m}{n} + \mathbb{Z}\right) = m + \mathbb{Z} = \mathbb{Z}$ with $\frac{m}{n} \in \mathbb{Q}$ and $n \in \mathbb{Z}$. By [11, 10.7] we can write $\mathbb{Q}/\mathbb{Z} = \oplus\ (\mathbb{Q}/\mathbb{Z})_p$ this means that

$$\mathbb{Z} = p^k\left(\frac{m}{n} + \mathbb{Z}\right) = p^k \frac{m}{n} + \mathbb{Z}$$

$$p^k\ \frac{m}{n} \in \mathbb{Z} \text{ if and only if } n\ /p^k \text{ if and only if } n = p^s \quad s \leq k$$

**Definition 3.05** A structure of the form $\{\frac{m}{p^s} + \mathbb{Z} \mid m \in \mathbb{Z}, s \in \mathbb{Z}^+\}$ is called a quasi-cyclic group and is denoted by $\mathbb{Z}_{P^\infty}$.

Now let us denote $c_n = \frac{1}{P^n} + \mathbb{Z}$ and consider the following:- $c_1 = \frac{1}{P} + \mathbb{Z}$, and $pc_1 = 0 \quad c_2 = \frac{1}{P^2} + \mathbb{Z}$, and $pc_2 = c_1$

In that order, we can write $pc_{n+1} = c_n$, by observing the nature of the elements.

Now for all $y \in \mathbb{Z}_{P^\infty}$, then $y \in mC_n$ therfore $\mathbb{Z}_{P^\infty}$ is generated by elements $c_1, c_2 - --, c_n$, where $pc_1 = 0 \left(of \ {}^{\mathbb{Q}}/_{\mathbb{Z}}\right)$, $pc_2 = c_1$ and so on.

Lastly, $\mathbb{Z}_{P^\infty} = \bigcup_{n=1}^{\infty} < c_n >$, and $\mathbb{Z}_{P^n} \cong < c_n >$.

**Theorem 3.1.4** [11, 10.13] There is an infinite $p$ -primary group $G = \mathbb{Z}_{P^\infty}$ each of whose proper subgroup is finite and cyclic.

Proof. Define a group $G$ having Generators $X = \{x_1, x_2, \dots \dots, x_n, \dots \dots\}$ and the relations $\{px_1, x_1 - px_2, \dots \dots, x_n - px_{n+1}\}$. Let $F$ be the free abelian group on $X$, let $R \leq F$ be generated by the relations, and let $a_n = x_n + R \in {}^{F}/_{R}$ = G. Then $pa_0 = 0$ and $a_{n-1} = pa_n$ for all $n \geq 1$, so that $p^{n+1}a_n = pa_0 = 0$. It follows that G is $p$ -primary, for $p^{t+1}\sum_{n=0}^{t} m_n a_n = 0$, where $m_n \in \mathbb{Z}$, A typical relation (i.e., a typical element of $R$) has the form: $m_n p x_0 + \sum_{n\geq 1} m_n(x_{n-1} - px_n) = (m_0 p - m_1)x_0 + \sum_{n\geq 1}(m_{n+1} - m_n p)x_n$.

If $a_0 = 0$, then $x_0 \in R$, and independence of $X$ gives the equations $1 = m_0 p + m_1$ and $m_{n+1} = pm_n$ for all $n \geq 1$. Since $R \leq F$ and $F$ is a direct sum, $m_n = 0$ for large $n$. But $m_{n+1} = p^n m_1$ for all $n$, and so $m_1 = 0$ Therefore, $1 = m_0\, p$, and this contradicts $p \geq 2$. A similar argument shows that $a_n \neq 0$ for all $n$. We now show that all $a_n$ are distinct, which will show that $G$ is infinite. If $a_n = a_k$ for $k > n$, then $a_{n-1} = pa_n$ implies $a_k = p^{k-n}a_n$, and this gives $(1 - p^{k-n})\, a_k = 0$; since $G$ is $p$-primary, this contradicts $a_k \neq 0$.

Let $H \leq G$. If $H$ contains infinitely many $a_n$, then it contains all of them, and $H = G$. If $H$ involves only $a_1, \dots, a_m$, then $H \leq < a_1, \dots \dots \dots . a_m > \leq < a_m >$.Thus, $H$ is a subgroup of a finite cyclic group, and hence $H$ is also a finite cyclic group.

## 3.2 Free Abelian Group

This is another very important notion of an abelian group theory; the idea of free abelian group is similar to that of vector spaces that we know in linear algebra.

**Definition 3.06** An abelian group $F$ is free abelian if it is a direct sum of an infinite cyclic groups.

The group is denoted as $F = \oplus < x >$, Thus $F$ consist of all linear combination of elements of $X$ as $g = n_1 x_{i_1} + n_2 x_{i_2} + --- + n_k x_{i_k}$.

Following the above definition we can say further that each $< x > \cong Z$ since each $< x >$ is infinite cyclic group, so now one can write $F = \sum_{x \in X} < x > = \sum \mathbb{Z}$. The elements of $X$ are called basis of $F$, if $X = \emptyset$ then $F = 0$

**Lemma 3.2.1** [11, 10.6] A set $X$ of nonzero elements of a group $G$ is independent if and only if $< X > = \oplus < x >$

Proof: Assume that $X$ is independent. If $x_0 \in X$ and $y \in < x_0 > \cap < X - \{x_0\} >$ Then $y = mx_0$ and $y = \sum m_i x_i$, where the $x_i$ are distinct elements of $X$ not equal to $x_0$. Hence $mx_0 = \sum m_i x_i$ *and* $mx_0 - \sum m_i x_i = 0$ so that independence gives each term 0; in particular, 0$= mx_0 = y$.

Conversely, let $\sum m_x x = 0$ for each $x \in X$, then $m_x x = \sum_{y \neq x} (-m_y) y \in < x_0 > \cap < X - \{x_0\} >$, $m_x x = 0$ this means that $X$ is independent.

**Lemma 3.2.2** [11, 10.4] let $\{A_k | k \in K\}$ be a family of subgroups of a group $G$. Then the following statements are equivalent.

i) $G \cong \oplus A_k$

ii) Every $g \in G$ has a unique expression of the form $g = \sum_{k \in K} a_k$

Proof: $(i) \to (ii)$ let $g \in G$ then let $g = \sum_{i \in I} a_i$ and $g \in \sum_{i \in I} {a_i}^1$

Then $g = \sum_{i \in I} a_i = \sum_{i \in I} {a_i}^1$ therefore we can write

$a_i - {a_i}^1 = \sum_{i \neq j} ({a_i}^1 - a_1) \in A_i \cap \sum A_j = 0$. Then $a_i = {a_i}^1$ Therefore $g$ is unique

$(ii) \to (i)$ For all $g \in G$, $g = \sum_{i \in I} a_i \in \sum A_i$ this means that $G = \sum A_i$

Next is to show that $A_i \cap \sum A_j = 0$, then let $a \in A_i \cap \sum A_j$

$a_i = a = a_{j_1} + a_{j_2} + \ldots + a_{j_k}$ where $j_1, \ldots \ldots \ldots \ldots j_k \neq i$, then by uniqueness $a_i = 0$ this means that $a = 0$ and $A_i \cap \sum A_j = 0$

Following above lemma we can clearly see that, if $X$ is a basis of a free group abelian group $F$, then each $\alpha \in F$ has a unique representation of the form $\alpha = \sum m_x x$, where $m_x \in \mathbb{Z}$ and $m_x \neq 0$ for only finite number of $x$ and zero otherwise. $X$ is independent by **lemma 3.2.1**

**Theorem 3.2.3** [11, 10.11] Let $F$ be a free abelian group with the basis $X$ and let $f: X \longrightarrow G$ be any function. Then there is a unique homomorphism $\varphi: F \longrightarrow G$, extending $f$ that is $\varphi(x) = f(x)$ for all $x \in X$, and if say $u = \sum m_x x \in F$, then $\varphi(u) = \sum m_x f(x)$.

$$\begin{tikzcd}
F \arrow[drr, "\varphi"] & & \\
X \arrow[u, dash] \arrow[rr, "f"] & & G
\end{tikzcd}$$

$fig$ 3.1

*Proof*

Assume that $X$ is independent. If $x_0 \in X$ and $y \in < x_0 > \cap < X - \{x_0\} >$. Then $y = mx_0$ and $y = \sum m_i x_i$, where the $x_i$ are distinct elements of $X$ not equal to $x_0$ Hence $mx_0 = \sum m_i x_i$ and $mx_0 - \sum m_i x_i = 0$ so that independence gives each term 0; in particular, 0= $mx_0 = y$.

If $u \in F$ ,then uniqueness of the expression $u = \sum m_x x$ shows that $\varphi: u \longrightarrow \sum m_x f(u)$ is a well defined function. That is $\varphi$ is a homomorphism extending $f$, it is obvious that $\varphi$ is unique because homomorphism agreeing on a set of generators must be equal.

Here is a fancy proof. For each $x \in X$, we know that there is a unique homomorphism $\varphi_x: < x > \longrightarrow G$ defined by $mx \longrightarrow mf(x)$ . The result now follows from **lemma 3.2.2** and by [11. 10.10]

<u>Note that</u>

(1) $X \subseteq F$

(2) $\varphi(u) = \sum m_x f(x)$. is an onto map, therefore by first Isomorphism theorem, we have;

(3) $G \cong {}^{F}/_{ker\varphi}$

The following corollary is immediate result of the above theorem.

**Corollary 3.2.4** [11, 10.12] Every abelian group $G$ is a quotient of free abelian group.

*Proof:* Let $F$ be the direct sum of $|G|$ copies of $\mathbb{Z}$, and let $x_g$ denote a generator of the $g^{th}$ copy of $\mathbb{Z}$ , where $g \in G$. Of course, $F$ is a free abelian group with basis $X = \{x_g : \in G\}$. Define a function $f: X \longrightarrow G$ by $f(x_g) = g$ for all $g \in G$. By **theorem 3.2.3** there is a homomorphism $\varphi: F \longrightarrow G$ extending $f$.

Now $\varphi$ is surjective, because $f$ is subjective, and $G \cong {}^{F}/_{ker\varphi}$ as desired. •

Following this we can recalled that our quasi-cyclic group can be generated by

a free-group $F < x_1, x_2, \dots \dots \dots \dots . >$ with the kernel $= \{px_1, x_1 - px_2 \dots , , x_n - px_{n+1}\}$.

**Example 3.1** Let $F =< x >$ and $G = \mathbb{Z}_5$ and define $\varphi: < x > \longrightarrow \mathbb{Z}_5$, let $f(x) = 1$ and $\varphi(nx) = n. f(x) = n. 1 = n$, then $ker\varphi =< 5x >$ and by 3 above $\mathbb{Z}_5 \cong {}^{< x >}/_{< 5x >}$

**Definition 3.07** The rank of a free abelian group is the cardinality of its basis. **Example 3.2** Let $X = \{x_1, x_2, x_3\}$ be the basis of a free group $F$, then each element $a \in F$ is of the form $a = m_1x_1 + m_2x_2 + m_3x_3 = \sum_{i=1}^{3} m_i x_i$ with each $< x_i > \cong Z$. This means that, this free abelian group is of rank three and can be written as $F = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

**Proposition 3.2.5** [9, 14.1] Free groups $F_M \; and \; G_N$ are isomorphic if and only if $M = N$, where $M$ and $N$ are the cardinality of the basis of respective groups.

Proof: Suppose that $M = N$ then there is a one-to-one function $f: M \longrightarrow N$ onto $N$ and defined $\varphi \; by \quad \varphi : F \longrightarrow G \quad \varphi(\sum_{m\in M} a_m m) = \sum_{m\in M} a_m f(m)$ clearly we can see that, $\varphi$ is one-to-one and onto, since $f$ is onto, therefore $F \cong G$

Suppose that $F \cong G$ and defined $\varphi : F \longrightarrow G$, then ${}^{F}/_{P^F} = \{a + P^F / \ a \in F\}$ with $P^F = \{Pa \ / \ \ a \in F\}$, is a field over $\mathbb{Z}_p$, also for the same reason ${}^{G}/_{PG}$. So ${}^{F}/_{P^F}$ and ${}^{G}/_{PG}$ are vector space over $\mathbb{Z}_p$ with the basis $\bar{M} =< m + P^F >$ and $\bar{N} =< n + P^G >$ and $|M| = |\bar{M}| = dim\, {}^{F}/_{P^F} = dim\, {}^{F}/_{PG} = |\bar{N}| = |N|$.

**Theorem 3.2.6** [9, 14.2] A set $X = \{x_i\}_{i\in I}$ of generators of a group $F$ is a free set of generators if and only if every mapping $\varphi$ of $X$ into a group $A$ can be extended to a (unique) homomorphism $\mu: F \longrightarrow A$.

Proof: Let $X$ be a free set of generators of $F$**.** If $\varphi : x_i \longrightarrow a_i$ is a mapping of $X$ into a group $A$ **,** then define $\mu: F \longrightarrow A$ as $\mu\big(n_1 x_{i_1} + \ldots\ldots\ldots\ldots\ldots\ldots. + n_i x_{i_k}\big) = n_1 a_{i_1} + \ \ldots\ldots\ldots\ldots\ldots. + n_1 a_{i_1}$, The uniqueness of **theorem 3.2.2** (ii) guarantees that $\mu$ is well defined, and it is readily checked that it preserves addition.

Conversely, assume that the subset $X$ in $F$ has the stated property. Then let $G$ be a free group with a free set $\{x_i\}_{i\in I}$of generators, where $I$ is the same as for $X$ . By hypothesis, $: x_i \longrightarrow a_i$ i $(i \in I\ )$ can be lifted to a homomorphism $\mu : \ F \longrightarrow G$, which cannot be anything else than the map $\mu: n_1 x_{i_1} + \cdots \ldots\ldots\ldots.. + n_i x_{i_k} \longrightarrow n_1 y_{i_1} + \ \ldots\ldots\ldots\ldots. + n_1 y_{i_1}$ **,** . It is evident that $\mu$ must be an isomorphism.

**Corollary 3.2.7** [9, 14.3] Every group with at most $m$ generators is an epimorphic image of free group $F$ of rank $m$

Proof: For an infinite cardinal $m$**,** $F_m$, has $2^m$ subsets, and hence at most $2^m$ subgroups and quotient groups. We infer that there exist at most $2^m$ pairwise nonisomorphic groups of cardinality $\leq m$ .

**Theorem 3.2.8** [9, 14.5] A subgroup of a free abelian group is free abelian.

Proof: Let $F = \bigoplus_{i\,\in I} < a_i >$ be a free group, and suppose that the index set $I$ is well ordered in some way; moreover, $I$ is the set of ordinals $< \tau$. For $\sigma \leq \tau$, we define $F_\sigma = \bigoplus_{p<\,\sigma} < a_p >$ If $G$ is a subgroup of $F$, then set $G_\sigma = G \ \cap F_\sigma$. Clearly, $G_\sigma = G_{\sigma+1} \cap F_\sigma$ so ${}^{G_{\sigma+1}}/_{G_\sigma} = {}^{G_{\sigma+1}}/_{G_{\sigma+1} \,\cap\, F_\sigma} \cong {}^{(G_{\sigma+1} + F_\sigma)}/_{F_\sigma}$. The latter quotient group is a subgroup of $F_{\sigma+1}/F_{\sigma} \cong < a_\sigma >$, thus either $G_{\sigma+1} = G_\sigma$ or $G_{\sigma+1}/G_{\sigma}$ is an infinite cyclic group. By [11, 14.4] we have $G_{\sigma+1} = G_\sigma \oplus < b_\sigma >$ for some $b \in G_{\sigma+1}$ (which is 0 if $G_{\sigma+1} = G_\sigma$). It follows that the element $b$ generates the direct sum $\oplus < b_\sigma >$. This direct sum must be $G$, because $G$ is the union of the $G_\sigma$.

Remember that in chapter two, we defined the notion of projectivity, at this time we can connect it to the notion of free Abelian group.

**Theorem 3.2.9** [9, 14.6] A group is projective if and only if it is free group.

Proof: Let $\beta: B \longrightarrow C$ be an epimorphism and $F$ a free group with $\varphi: F \longrightarrow C$**.** For each $x_i$ in a free set $\{x_i\}_{i \in I}$ of generators of $F$, we pick out some $b_i \in B$ such that $\beta b_i = \varphi x_i$ which is possible, since $\beta$ epic. The correspondence $x_i \longrightarrow b_i$ (i $\in I$ ) can, due to **theorem 3.2.6** be extended to a homomorphism $\mu: F \longrightarrow B$.This $\mu$ satisfies $\beta\mu = \varphi$; thus $F$ is projective.
Let $G$ be projective and $\beta : F \longrightarrow G$ an epimorphism of a free group $F$ upon $G$. Then there exists a homomorphism $\mu$: $G \longrightarrow F$ such that $\beta\mu = 1_G$. Hence $\mu$ is a monomorphism onto a direct summand of $F$, that is, $G$ is isomorphic to a direct summand of $F$. By **theorem 3.2.8**, $G$ is free.
Corollary 3.2.10 [11. 10.16] If $H \leq G$ and $G/H$ is free then $H$ is a direct summand of $H$. That is $G = H \oplus K$ where $K \leq G$ and $K \cong G/H$.

*Proof.* Let $F = G/H$ and let $\beta: G \to F$ be the natural map. Consider the diagram

$$\begin{tikzcd}
 & F \arrow[dl, "\gamma"'] & \\
G \arrow[r, "\beta"] & F \arrow[u, "1_F"'] \arrow[r] & 0
\end{tikzcd}$$

$fig$ 3.2

where $1_F$ is the identity map. Since $F$ has the projective property, there is a homomorphism $\gamma: F \longrightarrow G$ with $\beta\gamma = 1_F$. Define $K = Im\,\gamma$. The equivalence of (i) and (iii) in **Lemma 2.3.1** gives $G = ker\beta \oplus Im\gamma = H \oplus K$.

**Theorem 3.2.11** [11, 10.17] Every subgroup $H$ of a free abelian group $F$ of finite rank $n$ is itself a free abelian moreover, $rank(H) \leq rank(F)$.

*Proof*. The proof is by induction on $n$. If $n = 1$, then $F \cong \mathbb{Z}$, Since every subgroup $H$ of a cyclic group is cyclic, either $H = 0$ or $H \cong \mathbb{Z}$, and so $H$ is free abelian of rank $\leq 1$. For the inductive step, let $\{x_1, x_2 \dots \dots .., x_n, x_{n+1}\}$ be a basis of $F$. Define $F' = < x_1, \dots \dots .., x_n >$ and $H' = H \cap F'$. By induction, $H'$ is free abelian of rank $\leq n$. Now $H/_{H'} = H/_{(H \cap F')} \cong (H + F')/_{F'} \leq F/_{F'} \cong \mathbb{Z}$. By the base step, either $H/_{H'} = 0$ or $H/_{H'} \cong \mathbb{Z}$. In the first case, $H = H'$ and we are done; in the second case, Corollary 3.2.10 gives $H = H' \oplus < h >$, for some $h \in H$, where $< h > \cong \mathbb{Z}$, and so $H$ is free abelian and rank($H$) = rank($H' \oplus \mathbb{Z}$) = rank($H'$) + 1 $\leq n + 1$.

3.3 Finitely Generated Abelian Group

It is very important to note that every finite cyclic group is finitely generated, but there are infinite finitely generated abelian groups (Take $\mathbb{Z}$ for example)

**Definition 3.08** A group $G$ is finitely generated, if $G = < x_1, x_2, \dots \dots \dots .., x_n >$ that is for all $x \in G; x = \sum_{i=1}^{n} k_i x_i$.

**Theorem 3.3.1**[11, 10.19] Every torsion-free finitely generated group is free abelian

Proof: We prove the theorem by induction on $n$, where $G = < x_1, \dots, x_n >$. If $n = 1$ and $G \neq 0$, then $G$ is cyclic; $G \cong \mathbb{Z}$ (because it is torsion-free). Define $H = \{g \in G | mg \in < x_n > \text{ for some positive integer } m\}$. Now $H$ is a subgroup of $G$ and $G/_H$ is torsion-free: if $x \in G$ and $k(x + H) = 0$, then $kx \in H$ ,therefore $m(kx) \in < x_n >$, and so $x \in H$. Since $G/_H$ is a torsion-free group that can be generated by fewer than $n$ elements, it is free abelian, by induction. By **Corollary 3.2.10**, $G = F \oplus H$, where $F \cong G/_H$, and so it suffices to prove that $H$ *is* cyclic.

Note that $H$ is finitely generated, being a direct summand (and hence a quotient) of the finitely generated group $G$. If $g \in H$ and $g \neq 0$, then $mg = kx_n$ for some nonzero integers $m$ and $k$. It is easy to check that the function $\varphi: H \longrightarrow \mathbb{Q}$, given by $g \longrightarrow \frac{k}{m}$, is a well defined injective homomorphism; that is, $H$ is (isomorphic to) a finitely generated subgroup of $\mathbb{Q}$, say, $H = < {}^{a_1}/_{b_1}, \dots, {}^{a_i}/_{b_i} >$. If $b = \prod_i^n b_i$, then the map $\delta: H \longrightarrow \mathbb{Z}$ , given by $h \longrightarrow bh$, is an injection (because $H$ is torsion-free). Therefore, $H$ is isomorphic to a nonzero subgroup of $\mathbb{Z}$ , and hence it is infinite cyclic.

**Lemma 3.3.2** [9, 15.1] Let $G$ be a $p - group$ and assume that $G$ contain an element $g$ of maximal order $p^k$. Then $< g >$ is a direct summand of $G$.

**Theorem 3.3.3** [11, 15.1] The following statement on a group $G$ are equivalent.

(i) $G$ is finitely generated

(ii) $G$ is the direct sum of a finite number of cyclic groups;

(iii) The subgroups of $G$ satisfy the maximum order condition.

3.4 Divisible Group

We have seen a free group in which a connection between a free group and projective group was treated; in this section we shall see another connection between a divisible group and the dual of projectivity that is injective group.

**Definition 3.09** Let $G$ be a group and $g \in G$ which $0 \neq n \in \mathbb{Z}$, we say $g$ is divisible by $n$ if there is $a \in G$ with $g = na$ and denoted as $n|g$.

If all element of $G$ are divisible by every nonzero integer, then we say $G$ is divisible group

**Example 3.3** The following are divisible and non divisible groups

$(i)$ $\mathbb{Q}$ is a divisible group [since for every $q \in \mathbb{Q}$ we can write $m = \frac{p}{q} = n\left(\frac{p}{qn}\right)$] with $n \in \mathbb{Z}$ this means $n|m$ for all $m \in \mathbb{Q}$.

$(ii)$ $\mathbb{R}$ and $\mathbb{C}$ are divisible groups (For the same reason as above)

$(iii)$ ${}^{\mathbb{Q}}/_{\mathbb{Z}}$ is a divisible group (we shall see later)

$(iv)$ $\mathbb{Z}_{p^\infty}$ is a divisible group (since $\mathbb{Q}/\mathbb{Z}$ is divisible).

(v) $\mathbb{Z}$ is not divisible group (for example 1 is not divisible by 2 )

(v$i$) $\mathbb{Z}_n$ is not a divisible group (1 is not divisible by $n$)

From the above example we can easily see the following

(i) All infinite cyclic groups are not divisible since $\mathbb{Z}$ is not divisible.

(ii) All finite cyclic groups are not divisible since $\mathbb{Z}_n$ is not divisible.

(iii) If G is torsion free group then $g = na$ has at most one solution.

PROPERTIES OF DIVISIBILITY

(1) If $n|a$ and $n|b$ then $n|(a \pm b)$

(2) If $n|a$ , $m|a$ and $g\text{cd}(m,n) = 1$ then $mn|a$

(3) If $|a$ , $m|a$ and $g\text{cd}(m,n) \neq 1$ then $lcm[m,n]|a$ or $\frac{mn}{d}|a$ where d is the common divisor of m and n.

(4) If $f: G \longrightarrow H$ is a homomorphism and $n|\ a \in G$, then $n|f(a) \in H$

(5) If $o(a) = n\ and\ the\ \ g\text{cd}(m,n) = 1, then\ \ m|a.$

(6) If G is a direct sum that is $G = H \oplus K$, then $n|g = h + k$ $(h \in H, k \in K)$ if and only if $n|h$ and $n|k$ [this is due to the property (1) above]

**Proposition 3.4.1** [1] A homomorphic image of a divisible group is divisible.

Proof: Let $G$ be a divisible group and $f: G \longrightarrow H$, we claim $f(G)$ is divisible for all $h \in f(G)$, we can write $h = f(g)$ and since G is divisible for $g \in G$ and $0 \neq n \in \mathbb{Z}$ , we have $g = ng'$ with $g' \in G$ , therefore $h = f(g) = f(ng') = nf(g')$, this means that $n \mid h$.

Following above proposition, we can therefore state the corollary below:-

**Corollary 3.4.2** [1] If $G$ is divisible then for any subgroup $H$, then the quotient group $G/_H$ is divisible.

Proof: Since $G$ is divisible Take a canonical epimorphism $\sigma: G \longrightarrow {}^{G}/_{H}$. By **proposition 3.4.1**, $\sigma(G)$ is divisible and since $\sigma$ is an epimorphic then $\sigma(G) = {}^{G}/_{H}$, this means that ${}^{G}/_{H}$ is divisible.

Remember that we say ${}^{\mathbb{Q}}/_{\mathbb{Z}}$ is divisible because $\mathbb{Q}$ is divisible (Corollary 3.4.2)

**Proposition 3.4.3** [1] Direct sum (product) and direct summand of a divisible group is divisible.

Proof: $= \prod_{i\in I} D_i$, where each $D_i$ is divisible. Take $d_i \in G$ this means for $0 \neq n \in \mathbb{Z}$ we have $d_i = nd'_i$ with $d'_i \in G$ then $(d_i) = (nd'_i) = n(d'_i)$•

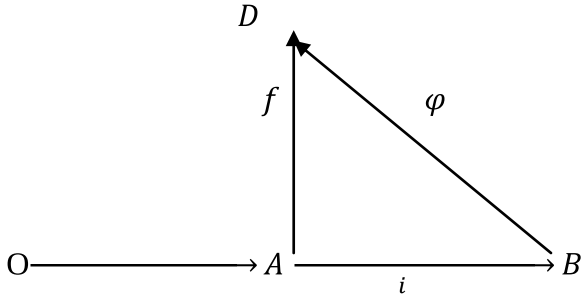
Let $G = H \oplus K$ and defined $P(G) = H$ where $P$ is the projection of the first coordinate $P(h, k) = h$ then by proposition 3.4.1 $H$ is divisible•

Let $G = \bigoplus_{i\in I} D_i$ and let $d_i \in G$ then for $0 \neq n \in \mathbb{Z}$, $(d_i) = n(d'_i)$ for some $d'_i \in \prod_{i\in I} D_i$. Now $d_i = 0 \; if \; i \in F$ where $F$ is some finite subsets of $I$.

Let $C_i$ be defined by $c_i = \begin{cases} b_i \; if \;\; i \in F \\ 0 \;\; if \;\; i \in F \end{cases}$

Now claim $(d_i) = (nc_i)$ that is $nc_i = d_i$. If $i \in F$ then $c_i = b_i$ this means that $d_i = nc_i$, if $i \in F$ then $c_i = 0$, this means $d_i = 0 = nc_i$, then $d_i = nc_i$.

**Theorem 3.4.4** [(11, 10.23) Baer, 1940, Injective property]:- let $D$ be a divisible group and let A be a subgroup of a group $B$. If $f: A \longrightarrow D$ is a homomorphism, then $f$ can be extended to a homomorphism $\varphi: B \longrightarrow D$. that is the following diagram commutes.



$fig$ 3.3

*Proof*. We use Zorn's lemma [9, page 2]. Consider the set $\mathfrak{D}$ of all pairs ($S$, $h$), where $A \leq S \leq B$ and $h$: $S \longrightarrow D$ is a homomorphism with $h_{/A} = f$. Note that

$\mathfrak{D} \neq 0$ Because $(A, f) \in \mathfrak{D}$. Then partially order $\mathfrak{D}$ by taking that $(S, h) \leq (S', h')$ if $S \leq S'$ and $h'$ extends $h$; that is, $h'lS = h$. If $\mathfrak{P} = \{(S_\alpha, h_\alpha)\}$ is a simply ordered subset of $\mathfrak{D}$ , define $(\tilde{S}, \tilde{h})$ by $\tilde{S} = \cup_\alpha S_\alpha$ and $\tilde{h} = \cup_\alpha h_\alpha$ ( this makes sense if one realizes that a function *is* a graph; in concrete terms, if $s \in S$, then $s \in S_\alpha$ for some $\alpha$, and $h(s) = h_\alpha(s)$ ). One can see that $(S, h) \in \mathfrak{D}$ and that it is an upper bound of $\mathfrak{P}$ '. By Zorn's lemma[9, page 2], there exists a maximal pair $(M, g) \in \mathfrak{D}$.

We now show that M = $B$, and this will complete the proof. Suppose that there is $b \in B$ with $b \notin$ M. If $M' = (M, b)$, then $M < M'$, and so it therefore suffices to define $h'\colon M' \longrightarrow$ D extending $g$ to reach a contradiction.

*Case* 1. $M \cap < b > = 0$.

In this case, $M' = M' \oplus < b >$, and one can define $h'$ as the map $m + kb \longrightarrow g(m)$.

*Case* 2. . $\mathrm{M} \cap < b > \neq 0$.

If $k$ is the smallest positive integer for which $kb \in M$, then each $y \in M'$ has a unique expression of the form $y = m + tb$, where $0 < t < k$. Since $D$ is divisible, there is an element $d \in D$ with $kd = h(kb)$ ($kb \in M$ implies $h(kb)$ is defined). Define $h'\colon M' \longrightarrow D$ by $m + tb \longrightarrow g(m) + tb$. One can easily see that, $h'$ is a homomorphism extending $g$

The following result is the immediate consequences of the above theorem

**Theorem 3.4.5** [9, 21.2 (Baer)] If a divisible group $D$ is a subgroup of a group $G$, then $D$ is a direct summand.

Proof: Consider the diagram below

$$\begin{tikzcd}
& D & \\
0 \arrow[r] & D \arrow[u, "1_D"] \arrow[r, "i"'] & G \arrow[ul, "\varphi"']
\end{tikzcd}$$

$fig$ 3.4

where $1_D$ is the identity map. By the injective property, there is a homomorphism $\varphi\colon G \longrightarrow D$ with $\varphi_i = 1_D$ (where $i$ is the inclusion map from $D$ to $G$); that is, $\varphi_i(d) = d$ for all $d \in D$. By Lemma 3.2.6, $D$ is a direct summand of $G$.

**Lemma 3.4.6** [1] The sum of any family of divisible groups is divisible.

Proof: Let $G = \sum_{i\in I} D_i = \{\sum d_i : d_i = 0$ for all $i$ except for finite number of $i\}$ Take $g \in G$ and $0 \neq n \in \mathbb{Z}$, then $g = \sum d_i$ *and* since each $D_i$ is divisible and $d_i = nd'_i$ for some $d'_i \in D_i$, therefore $g = \sum d_i = \sum nd'_i = n\sum d'_i = ng'$.

**Definition 3.10** If $G$ is a group then $dG$ is a subgroup generated by all divisible subgroups of $G$ and is called the divisible part of $G$.

**Definition 3.11** A subgroup $H$ of a group $G$ is fully invariant if $f(H) \subseteq H$.

Note that $dG$ is a fully invariant subgroup of $G$ (since image of divisible group is divisible)

**Definition 3.12** A group $G$ is reduced if $dG = 0$

**Theorem 3.4.7**[9, 21.3] For every group $G$ there is a decomposition $G = dG \oplus R$, where $R$ is reduced.

Proof: Assume that $G = D' \oplus R'$ , Here $dG$ *is* a uniquely determined subgroup of $G$, while $R$ is unique up to isomorphism. The fact that $dG$ is the maximal divisible subgroup of $G$ and $G = D' \oplus R'$, where $D'$ is divisible and $R'$ as reduced, then $D' \leq dG$, and by [9, 9.3] we have $dG = (dG \cap D' ) \oplus (dG \cap C')$. Note that $(dG \cap C') = 0$ as a direct summand of a divisible group contained in a reduced group, thus $dG \cap D' = dG$, then we can write $dG \leq D'$, and so $dG = D'$.

**Definition 3.13** A group G is called $p-$ divisible $if$ $p^kD = D$ for every positive integer $k$.

Remember that we can write $p^kD = p.\dots pD,$ it is obvious that $p-$divisibility implies divisibility.

Recall that for any group $G$ and $0 \neq n \in \mathbb{Z}$, then $nG = \{na \mid a \in G\}$ and $G[n] = \{a \in G \mid na = 0\}$

**Lemma 3.4.8** [11, 10. 27] If $G$ and $H$ are divisible $p-$ primary groups, then $G \cong H$ if and only if $G[p] \cong H[p]$.

Proof: Necessity follows easily from the fact that $\varphi(G[p]) \leq \varphi(H[p])$ for every homomorphism $\varphi: G \longrightarrow H$.

For sufficiency, assume that $\varphi$: *G[p]* $\longrightarrow$*H[p]* is an isomorphism; composing with the inclusion *H[p]* $\subseteq H$, we may assume that $\varphi$: *G[p]* $\longrightarrow$ *H*. The injective property gives the existence of a homomorphism $\psi$: G$\longrightarrow$ $H$ extending $\varphi$; we claim that $\psi$ is an isomorphism.

(i) $\psi$ is injective.

We show by induction on $n \geq 1$ that if $x \in G$ has order $p^n$, then $\psi$ (x) = 0. If $n =$ 1, then $x \in$ *G[p]*, so that $\psi$ (x) $= \varphi(x) = 0$ implies $x = 0$ (because $\varphi$ is injective). Assume that $x$ has order $p^{n+1}$ and $\psi$(x) = 0. Now $\psi(px) = 0$ and $px\ has\ order\ p^n$, so that $px = 0$, by induction, and this contradicts $x$ having order $p^{n+1}$

(ii) $\psi$ is surjective.

We show, by induction on $n \geq$1, that if $y \in H$ has order $p^n$, then $y \in$ im $\psi$, If $n$ = 1, then $y \in$ *H[p]* = im $\varphi \leq$ im $\psi$. Suppose now that y has order $p^{n+1}$, since $p^n y \in$ *H[p]*, there is $x \in$ $G$ with $\psi$ (x) = $p^n y$; since $G$ is divisible, there is $g \in G$ with $p^n g = x$. Thus, $p^n(y - \psi$(x)) = 0, so that induction provides $z \in$ G with $\psi$(z) = $y$ - $\psi(g)$. Therefore, $= \psi(z + g)$, , as desired.

**Lemma 3.4.9** [1] if G is divisible then tG is also divisible.

Proof: let $a \in tG \leq G, 0 \neq n \in \mathbb{Z}$, and $a = nb$ with $b \in G$ then $ma = 0$ , for some $m \neq 0$. But $(mn)b = ma = 0$ this means that $b \in tG$ therefore $tG$ is divisible

**Lemma 3.4.10**[1] For every group $G$ and prime $p$, $G[p]$ can be made a vector space over $\mathbb{Z}_p$

Proof: For $m, n \in \mathbb{Z}_p, we\ claim\ that\ (m + n)a = ma + na\ with\ a \in G[p]$

$m + n = p.q + r\ but\ m + n = r\ \in \mathbb{Z}_p, \qquad then\ ra = ma + na$

$ma + na = (m + n)a = (p.q + r)a = q(pa) + ra, but\ pa = 0\ then$

$ma + na = ra$

**Theorem 3.4.11** [11, 10.28] Every divisible group $G$ is a direct sum of copies of $\mathbb{Q}$ $and$ $\mathbb{Z}_{p^\infty}$ for various p.

Proof: $tG$ is a divisible subgroup of $G$ therefore $G = tG \oplus C,$ where $C$ is vector $space\ over$ $\mathbb{Q}$ therefore by [11, 10.5] $C\ is\ a\ direct\ sum\ of\ copies$ of $\mathbb{Q}$ then $C \cong \oplus\ \mathbb{Q}$, $tG = \oplus\ T_p(G)$ by $\mathbf{theorem\ 3.1.2}$ we know $T_p(G)$ is a $p-$ primary divisible group then by **lemma 3.4.10** $tG[p]$ can be made vector space over $\mathbb{Z}_p$ this means that $T_p(G)[p] \cong \oplus\ \mathbb{Z}_p$. Take $H = \oplus\mathbb{Z}_{p^\infty}$ we see that $H[p] = \oplus\ \mathbb{Z}_{p^\infty}[p] = \oplus < C_0 > \cong \oplus\ \mathbb{Z}_p$ then $H[p] \cong T_p(G)[p]$ $\mathbf{by\ theorem\ 3.4.8}$ $T_p(G) \cong H = \oplus\mathbb{Z}_{p^\infty}$ and this means that $G = \oplus\ (\oplus\mathbb{Z}_{p^\infty}) \oplus (\oplus\ \mathbb{Q})$ as required.

**Theorem 3.4.12** [11, 10.30] Every group $G$ can be imbedded in a divisible group.

***Proof.*** Write $G = F/R$, where $F$ is free abelian. Now $F = \sum \mathbb{Z}$, so that F $\leq \sum \mathbb{Q}$ (Just imbed each copy of $\mathbb{Z}$ into $\mathbb{Q}$). Hence $G = {}^{F}/_{R} = \sum {}^{F}/_{R} \leq \sum {}^{\mathbb{Q}}/_{R}$,

and the last group is divisible, being a quotient of a divisible group.

**Corollary 3.4.12** [11, 10.31] A group $G$ is divisible if and only if it is a direct summand of any group containing it.

Proof: Necessity is from the **theorem 3.4.5** that is if $H$ is divisible and $H \leq G$, then $G = H \oplus K$ For some $K \leq G$.

Sufficiency, Theorem 3.4.12 $H$ can be embedded in a divisible group $G$ $0 \longrightarrow H \xrightarrow{f} G$, then $H \cong Imf$ this means that $G = Imf \oplus K$, $Imf\ is$ divisible therefore $H$ is divisible.

## 3.5 Pure and Basic Subgroup

The notion of pure subgroup becomes one of the most useful concepts in abelian group theory. This notion is the intermediate between subgroups and direct summand. It is important to note that direct summand are always pure but the converse need not be true.

**Definition 3.14** A subgroup $H \leq G$ is pure in $G$ if $H \cap (nG) = nH$ for every integer n $> 0$. In other words every element $h \in H$ which is divisible by $n$ in $G$ must also be divisible by $n$ in $H$.

**Definition 3.15** A subgroup $H$ of $G$ is $p-pure$ (p = prime) if $H \cap (p^k G) = p^k H$ for k= 1, 2,.....................,

**Example 3.4** Every direct summand is pure.

Let $G = H \oplus K\ if\ h \in H$ and $h = ng$ with $g \in G,$ and claim $h = nh',\ h' \in H$ Take $g = h' + k'$then $h = n(h' + k')$ this means $h - nh' = nk' \in H \cap K = 0$ this means $h = nh'$ as claim.

**Example 3.5** If $H \leq G\ and\ {}^{G}/_{H}$ is torsion-free then $H$ is pure.

$h = ng$ then $n(g + H) = ng + H = h + H = H,$ so $g + H \in {}^{G}/_{H}$ has a finite order, but ${}^{G}/_{H}$ is torsion-free this means that $g + H = H$ and $g \in H.$

**Example 3.6** $tG$ is a pure subgroup of a group $G.$ (Note that this may not be a direct summand)

Remember **theorem 3.1.1** says ${}^{G}/_{tG}$ is torsion $-$ free and **example 3.5** says $tG$ is pure.

Following [11, 10.2] one can see that $tG$ may not be a direct summand of $G.$

**Example 3.7** Let $H = \bigoplus_{i \in I} A_i$ and $G = \prod_{i \in I} A_i$ then $H$ is a pure subgroup of $G.$

Really let $x = ny \in H$ with $y \in G$ where $x = (x_i)$ and $y = (y_i)$ let $F = \{i \in I: x_i \neq 0\}$ then F is finite, Take $y' = (y'_i)$ and defined $y' = \begin{cases} y_i & i \in F \\ 0 & i \notin F \end{cases}$

Note that $y' \in H.$ since $x = ny,$ we can write $x_i = ny_i$ for all $i \in I$

If $i \notin F$ then $ny'_i = 0 = x_i$ and if $i \in F$ then $y'_i = y_i$ and $x_i = ny_i = ny'_i,$ so $x = ny'.$

If $C \leq B \leq A$ then $B \cap (A + C) = (B \cap A) + C$ we can easily see this, since each side is in B. Now is the right time for the next lemma.

**Lemma 3.5.1** [9, 26.1] Let $B$ and $C$ be subgroups of an abelian group $G$ such that $C \leq B \leq G$ then we have

(i) If $C$ is pure in $B$ and $B$ is pure in $G$ then $C$ is pure in $G$ (Transitivity)

(ii) If $B$ is pure in $G$, then ${}^{B}/_{C}$ *is* pure in ${}^{G}/_{C}$.

(iii) If $C$ is pure in $G$ and ${}^{B}/_{C}$ *is* pure in ${}^{G}/_{C}$ then B is pure in $G$

Proof: (i) $nC = C \cap (nB) = C \cap \big(B \cap (nG)\big) = (C \cap B) \cap (nG) = C \cap (nG)$ for every n$>$ 0 proving the purity of C in $G$

(ii) $n({}^{B}/_{C}) = nB + C = \big(B \cap (nG)\big) = B \cap (nG + C) = B \cap n({}^{G}/_{C})$

For every n $\neq$ 0 proving the purity of ${}^{B}/_{C}$ in ${}^{G}/_{C}$

(iii) Let $ng = b \in B$ for some $g \in G$ and integer n$>$ 0 then we writes $n(g + C) = b + C$, since ${}^{B}/_{C}$ *is* pure in ${}^{G}/_{C}$ then for some $b' \in B$, we write $n(b' + C) = b + C$ and $nb' = b + c = ng + C(c \in C), then$ we have $n(b' - g) = c$, since $C$ is pure in $G$ thus $c = nc'$ for some $c' \in C$, then $n(b' - g) = c$, implies that we can write
$nb' - b = nc'$ this means that $b = n(b' - c')$, with $(b' - c') \in B$, thus $B$ is pure in $G$

**Lemma 3.5.2**[1] Let $H \leq M \leq G$ *and* $H$ *is pure in* $G$ *then* $M$ is pure in $G$ if and only if ${}^{M}/_{H}$ is pure in ${}^{G}/_{H}$

Proof: Necessity follows from **lemma 3.5.1** (ii) and sufficiency follows from the same lemma but (iii)

**Lemma 3.5.3**[11, 10.34] A $p$ -primary group $G$ that is not divisible contains a pure non-zero cyclic subgroup.

Proof: Assume first that there is $x \in G[p]$ that is divisible by $p^k$ but not by $p^{k+1}$, and let $x = p^k y$. we need to show that $< y >$ is pure in $G$. Let $my = na$ $a \in G$ *and take* $m = p^t m'$ where p $\nmid m'$*and* $n = p^s n'$*where* $p \nmid n'$,If $t \geq k + 1$ *then* $my = p^{t-k-1} m' p(p^k y) = 0$, suppose that $t \leq k$, we claim that for $x \in G$ *then* $t \geq s$, now assume that $t < s$ this means that $s \geq t + 1$, $m'x = m'p^k y = p^{k-t} p^t m' y = p^{k-t}(my) = p^{k-t}(na) = p^{k-t}(p^s n' a)$

$= p^{k-t+s} n' a = p^{s-t-1} p^{k+1} n' a = p^{k+1}(p^{s-t-1} n' a).$

$gcd(m', p) = 1$ *since* p $\nmid m'$*therefore, for* $u, v \in \mathbb{Z}$, *we have* $m'u + pv = 1$ and $x = um'x + vpx = um'x = p^{k+1}(up^{s-t-1} n' a)$, which means $p^{k+1}$ / $x$ contradicting the first assumption.

If $t \geq s$ this means $my = p^t my' = p^s\left(p^{t-s}(m'y)\right)$ and take b = $p^{t-s}(m'y)$, $g\mathrm{cd}(m', p) = 1$, since p $\nmid n'$then for $u, v \in \mathbb{Z}$, we have $n'u + pv = 1$, therefore b = $un'b + vpb = u(n'b)$ and $my = p^s n'(ub) = n(ub)$, but $ub = \mathrm{u}p^{t-s}m'y$ and $\mathrm{u}p^{t-s}m'y \in< y >$, this means that $< y >$ is pure in $G$.

We may, therefore, assume that every $x \in G[p]$ is divisible by every power of p. In this case, we prove by induction on $k \geq 1$ that if $x \in G$ and $p^k x = 0$,then $x$ is divisible by $p$. If $k$ = 1, then $x \in G[p]$, and the result holds. If $p^{k+1}x = 0$, then $x \in G[p]$, and so there is $z \in G$ with $p^{k+1}z = p^k x$ Hence $p^k(pz - x) = 0$. By induction, there is $x \in G$ with $pw = pz - x$ and $x = p(z - w)$, as desired.

**Definition 3.16** A subset $X$ of an abelian group $G$ is pure-independent if; it is independent and $< X >$ is a pure subgroup of $G$ ( see lemma 3.2.1 for condition of independency).

**Lemma 3.5.4** [11, 10.35] Let $G$ be a $p$ -primary group if $X$ is a maximal pure – independent subset of $G$, ${}^{G}/_{< X >}$ then divisible.

Proof: If ${}^{G}/_{< X >}$ is not divisible, then Lemma 3.5.3 shows that it contains a pure nonzero cyclic subgroup $< \bar{y} >$ and by [11, 10.32] we may assume that $y \in G$ and $\bar{y} \in {}^{G}/_{< X >}$ have the same order (where $y \longrightarrow \bar{y}$ under the natural map). We claim that {X,y} is pure-independent. Now $< X > \leq < X, y > \leq G$ and ${}^{< X,y >}/_{< X >} =< \bar{y} >$ is pure in ${}^{G}/_{< X >}$ by [11,10.32] $< X, y >$ is pure in $G$. Suppose that $my + \sum m_i x_i = 0$, where $x_i \in X$ and $m, m_i \in \mathbb{Z}$ In ${}^{G}/_{< X >}$, this equation becomes $m\bar{y}$ = 0. But $y$ and $\bar{y}$ have the same order, so that $my$ = 0.Hence $\sum m_i x_i$ = 0, and independence of $X$ gives $m_i x_i = 0$ for all $i$. Therefore $\{X, y\}$ is independent, and by the preceding paragraph, it is pure-independent, contradicting the maximality of $X$.

**Definition 3.17** A subgroup $B$ of a torsion group $G$ is a basic subgroup if;

(i) $B$ is a direct sum of cyclic groups;

(ii) $B$ is a pure subgroup of $G$; and

(iii) ${}^{G}/_{B}$ is divisible.

**Theorem 3.5.5** [13, 10.36] Every torsion group $G$ has a basic subgroup.

Proof: Let $G = \sum G_p$ be the primary decomposition of $G$. If $G_p$ has a basic Subgroup of $B_p$,then it is easy to see that $\sum B_p$ is a basic subgroup of $G$. Thus, we may assume that $G$ is p-primary. If $G$ is divisible, then $B$ = 0 is a basic subgroup. If $G$ is not divisible, then it contains a pure nonzero cyclic subgroup, by **Lemma 3.5.3**, that is, $G$ does have pure-independent subsets. Since both purity and independence are preserved by ascending unions, Zorn's lemma applies to show that there is a maximal pure-independent subset X of $G$. But **Lemma 3.2.1** and [11, 10.33] shows that $B = < X >$ is a basic subgroup.

**Corollary 3.5.6** [11, 10.37] If $G$ is a group of bounded order (that is $nG = 0$ for some n > 0) then $G$ is a direct sum of cyclic group.

Proof: $G$ is torsion by **theorem 3.5.5** $G$ has a basic subgroup $B$ and ${}^{G}/_{B}$ is divisible but $nG = 0$ therefore $n({}^{G}/_{B}) = 0$, now let $\bar{g} \in {}^{G}/_{B}$, then we can write $\bar{g} = n\bar{h}$ for some $\bar{h} \in {}^{G}/_{B}$ since $n({}^{G}/_{B}) = 0$ ; then $n\bar{h} = 0$ this means that $\bar{g} = 0$ which gives ${}^{G}/_{B} = 0$ $and$ $G = B$, but $B$ is a basic subgroup then $G = B = \bigoplus_{i \in I} < x_i >$.

**Corollary 3.5.7** [11, 10.41] A pure subgroup $H$ of bounded order is a direct summand.

Note with this we can now concentrate with the remaining few notions that will be presented in the next chapter before presenting the main work of this thesis

# CHAPTER FOUR

## 4.1 Small and Essential Subgroups

The notion of a small subgroup is the most useful notion of this research work, the basic idea of our thesis is due to a small subgroup and the well known notion of the homomorphism.

**Definition 4.01**[8] A subgroup $H$ of an abelian group $G$ is called small or (superfluous) in $G$ if for all subgroup $K$ of $G,$ then equality $H + K = G$ implies $K = G$.

Notation: if $H$ is a small subgroup of a group an abelian $G$ then we write $H \ll G$

From our definition we can obtain the following remark:-

1) $H \ll G$ if and only if for all $K < G$ implies that $H + K < G$
2) If $0 \neq G$ and $H \ll G$ then $H \neq G$ ( if $H = G$ then H + 0 = $G$, which means $G$ = 0 from the definition that contradict $G \neq 0$)

**Definition 4.02** A group $G$ is called a simple group, if $G$ has no non-trivial nonzero subgroup (That is $G$ has only 0 and itself as subgroups).

**Example 4.1)** For any group $G$, 0 is a small subgroup.

**Example 4.2)** A subgroup $< C_n >$ is a small subgroup of $\mathbb{Z}_{p^\infty}$ for each $n$.

**Example 4.3)** In $\mathbb{Z}$, 0 is the only small subgroup.

**Example 4.4)** for every simple group, 0 is the only small subgroup.

**Example 4.5)** In a free abelian group only the non-trivial subgroup 0 is small.

**Definition 4.03** A homomorphism $f: G \longrightarrow N$ is called a small homomorphism if K$erf \ll G$.

**Definition 4.04** [1] Let $G$ and $N$ be subgroups then the set of homomorphisms $\varphi: G \longrightarrow N$, Hom$(G, N)$ is a group of homomorphisms $\varphi: G \longrightarrow N$ with respect to operation $(\varphi + \beta)g = \varphi(g) + \beta(g)$.

**Lemma 4.1.1** [8, 5.1.3]

(a) If $A \leq B \leq C \leq G$ and $B \ll C$ then $A \ll G$.
(b) If $A_i \ll G, i = 1, \dots, n$ then $\sum_{i=1}^{n} A_i \ll G$.
(c) If $A \ll G$ and $\varphi \in Hom(G, N)$ then $\varphi(A) \ll N$.
(d) If $f: A \longrightarrow B$, and $\beta: B \longrightarrow C$ are small epimorphism then $\beta f: A \longrightarrow C$ is also a small epimorphism

**Definition 4.05** [8] A subgroup $H$ of a group $G$ is essential (large) in $G$, if for all subgroups $U \leq G$, $H \cap U = 0$ implies $U = 0$

Notation: if $H$ is an essential (large) subgroup of a group $G$ then we write $H \trianglelefteq G$.

**Definition 4.06** A homomorphism $f: A \longrightarrow B$ is called essential if $Imf \trianglelefteq B$.

From **definition 4.04** we can immediately obtain the following remark:-

1) $H \trianglelefteq G$ if and only if for all $U \leq G$ and $U \neq 0$ , $H \cap U \neq 0$
2) If $G \neq 0$ $and$ $H \trianglelefteq G$ then implies that $H \neq 0$

**Example 4.6)** Every non zero subgroup of $\mathbb{Z}$ is essential in $\mathbb{Z}$.

**Example 4.7)** $\mathbb{Z}$ is essential in $\mathbb{Q}$.

**Lemma 4.1.2** [8, 5.1.5]

(a) If $A \leq B \leq C \leq G$ and $A \trianglelefteq G$ then implies $B \trianglelefteq C$
(b) If $A_i \ll G, i = 1, \dots, n$ then $\bigcap_{i=1}^{n} A_i \trianglelefteq G$.
(c) If $B \trianglelefteq N$ and $\varphi \in Hom(G, N)$ then $\varphi^{-1}(B) \trianglelefteq G$.
(d) $f: A \longrightarrow B$, and $\beta: B \longrightarrow C$ are large homomorphisms then $\beta f: A \longrightarrow C$ is also a large homomorphism

## 4.2 Semisimple Group

**Theorem 4.2.1**[15, 8.1.3] for a group $G$ the following conditions are equivalent:

(1) Every subgroup of $G$ is a sum of simple groups
(2) $G$ is a sum of simple subgroups
(3) $G$ is a direct sum of simple subgroups.
(4) Every subgroup of $G$ is a direct summand of $G$

**Definition 4.07** A group $G$ which satisfy the condition of the **theorem 4.2.1** is called a semisimple group.

**Example 4.8)** A group $\mathbb{Z}/_{n\mathbb{Z}}$ with $n \neq 0$ is a semisimple abelian group if and only if n is a square-free integer or $n = \pm 1$.

**Example 4.9)** If $G = \oplus \mathbb{Z}_p$ and $p$ is prime then $G$ is a semisimple group.

**Example 4.10)** $\mathbb{Z}$ and $\mathbb{Q}$ are no semisimple abelian groups (they have no simple subgroup)

**Corollary 4.2.2** [8, 8.1.5] For a semisimple abelian group, we have;

1) Every subgroup of a semisimple group is semisimple.
2) Every epimorphic image of a semisimple group is semisimple.
3) Every sum of semisimple group is semisimple.

## 4.3 Radical of a Group

**Theorem 4.3.1**[8, 9.1.1] let $G$ be an abelian group, then $\sum_{A \ll G} A = \cap\, pG$, where $p$ is prime.

**Definition 4.08** The subgroup of $G$ defined by the theorem 4.3.1 is called the radical of a group and is denoted by Rad $G$.

**Theorem 4.3.1**[8, 9.1.4] for a group $Hom(G, N)$, we have the following:-

(a) If $\varphi \in Hom(G, N)$ then $\varphi(Rad\ G) \leq Rad\ N$;

(b) $Rad\left(G/_{Rad\ G}\right) = 0$ and for all $C \leq G, Rad\left(G/_{C}\right) = 0$ implies $Rad\ G \leq C$.

**Corollary 4.3.2** [8, 9.1.5] for all abelian group, we have the following:-

(a) Epimorphism $\varphi: G \longrightarrow N$ if ker $(\varphi) \ll G$, implies $\varphi(Rad\ G) = Rad\ N$ and $Rad\ G = \varphi^{-1}(Rad\ N)$.

(b) If $C \leq G,$ then $Rad\ C \leq Rad\ G$.

(c) If $G = \bigoplus_{i \in I} M_i$ then $Rad\ G = \bigoplus_{i \in I} Rad(M_i)$.

(d) If $G = \bigoplus_{i \in I} M_i$ then $G/_{Rad\ G} \cong \bigoplus_{i \in I} \left(M_i/_{Rad(M_i)}\right)$.

**Example 4.11)** $Rad(\mathbb{Z}) = 0$, since by definition 4.01, 0 is the only small subgroup of $\mathbb{Z}$.

**Example 4.12)** $Rad(\mathbb{Q}) = \mathbb{Q}$, since for every $q \in \mathbb{Q}$, $q\mathbb{Z}$ is small in $\mathbb{Q}$ by **definition 4.01** (also the same as saying $\mathbb{Q}$ has no maximal subgroup).

**Theorem 4.3.3** [8, 9.2.1]

(a) If $G$ is a semisimple abelian group then $Rad\ G = 0$,

(b) If $G$ is finitely generated then $Rad\ G \ll G$.

(c) If $G$ is finitely generated and $G \neq 0\ then\ Rad\ G \neq G$.

# CHAPTER FIVE

## 5.1 Characterization of $T$-noncosingular Abelian Groups

Throughout this chapter we will adopt $E$ to be the set of endomorphisms of an abelian group, Motivated by [Tutuncu and Tribak, 2009] and [Tribak, 2013] we present the notion of $T$-noncosingular Abelian group. An abelian group $G$ is $T$-noncosingular if for every nonzero endomorphism of $f$, the $Im\, f$ is not small in $G$. Following **definition 4.01** we can now define the concept $T$-noncosingular abelian group.

Following [Talebi and Vanaja, 2007] $G$ will be called noncosingular if for every nonzero homomorphism $f: G \longrightarrow H$, $Im\, f$ is not small in $H$.

**Definition 5.01** Let $G$ and $H$ be two Abelian groups. We say that $G$ is $T$-noncosingular relative to $H$, if for every $0 \neq f \in Hom(G,H)$, the $Im\, f$ is not small in $H$.

**Definition 5.02** Let $G$ be an abelian group. We say that $G$ is a $T$-noncosingular abelian group if it is $T$-noncosingular relative to itself, that is for every $0 \neq f \in E = End(G)$, the $Im\, f$ is not small in $G$. In other words $G$ is $T$-noncosingular if and only if for every nonzero endomorphism $f$ of E, $Im\, f \ll G$ implies that $f = 0$.

From the two definitions above we can clearly see that every noncosingular is also $T$-noncosingular Abelian group; however we can see that $\mathbb{Z}_p$ is $T$-noncosingular but not noncosingular which means the converse need not be true. Really for the nonzero endomorphism $f: \mathbb{Z}_p \longrightarrow \mathbb{Z}_{p^2}$, defined by $f(k) = pk$ we have $Imf = < p > \ll \mathbb{Z}_{p^2}$.

**Proposition 5.1.1** Every simple group $S$ is $T$-noncosingular.

Proof; For every $0 \neq f \in \mathrm{End(S)}$, $Im\, f \leq S$ and $S$ is simple this means that $Im\, f = S$. This means that $Im\, f$ is not small in $S$.

We already know that divisible groups are injective groups and the image of a divisible group is a direct summand from 3.4.5, we can now state the following.

**Proposition 5.1.2** Every divisible group $D$ is $T$-noncosingular.

Proof; for every $0 \neq f{:}D \longrightarrow D$, this means that $Im\, f$ is also divisible and hence a direct summand of $D$ this means $D = Im\, f \oplus K$ , for some subgroup $K$, this means that $Im\, f$ is not small in $D$.

Remember that for an abelian group $G$ the radical of a group $G$ is Rad $G = \cap\, pG$, where $p$ runs over all prime integers.

**Proposition 5.1.3** If Rad $G$ = 0 then $G$ is $T$-noncosingular.

Proof: suppose that $Im\, f \ll G$, for an endomorphism $f{:}\,\mathrm{G} \longrightarrow \mathrm{G}$. Then $Im\, f \leq$ Rad $G$ = 0, therefore. $Im\, f = 0$ that is $f = 0$ by **definition 5.02** and so $G$ is $T$-noncosingular.

From **example 4.11** we know that Rad $\mathbb{Z}$ = 0, we can state the corollary below;

**Proposition 5.1.4** $\mathbb{Z}$ is $T$- noncosingular abelian group.

Proof: follows from **proposition 5.1.3**

**Corollary 5.1.5** Let $A$=$\left\{\frac{n}{m} \in \mathbb{Q}{:}\, m\ is\ a\ square\ free, m = p_1.p_2 \ldots p_k\right\}$, then $A$ is $T$- noncosingular.

Proof: Let $f{:}\,\mathrm{A} \longrightarrow \mathrm{A}$ be an endomorphism with $Im\, f \ll A$ . Then $Im\, f \leq$ Rad $A = \cap\, pA = \mathbb{Z}$ where the intersection is taken over all prime numbers $p$. On the other hand $1 \in \mathbb{Z} =$ Rad $A$, therefore $f(1) \in f(Rad\ A) \subseteq$ Rad $f(A) =$ Rad $Im\, f \leq Rad\,\mathbb{Z} = 0$. So we see that $1 \in Kerf$, hence $\mathbb{Z} \subseteq Kerf$. Then for every $\frac{n}{m} \in A$, we have $mf\left(\frac{n}{m}\right) = f\left(m\frac{n}{m}\right) = f(n) = 0$. But A is torsion free, hence $f\left(\frac{n}{m}\right) = 0$. So $f = 0$.

We have been mentioning different abelian groups which are $T$-noncosingular, let us at this point state some useful examples.

**Example 5.1)** $\mathbb{Z}_{p^n}$ is not $T$-noncosingular abelian group, for any integer n $> 1$, and prime $p$.

Proof: Take $0 \neq f \in E =$End $(\mathbb{Z}_{p^n})$ defined by $f(k) = p^{n-1}k$, then this means that $Im\, f = \langle p^{n-1} \rangle \cong \mathbb{Z}_p$, but one can see that $\langle p^{n-1} \rangle \ll \mathbb{Z}_{p^n}$ .

**Proposition 5.1.6** [Tutuncu and Tribak 2009] Let $G$ be a $T$-noncosingular abelian group and $H$ be a direct summand of $G$, then $H$ is also $T$- noncosingular.

Proof:- Let $G = H \oplus K$ and define $f : H \longrightarrow H$ with $Im\, f \ll H$ then consider the homomorphism $g = f \oplus 0{:}\ G \longrightarrow G$ defined by $g(h,k) = (f(h),0)$. Then $Imf = Imf \oplus 0 \ll H \oplus K$. Since $G$ is *T-noncosingular,* $g = 0$ therefore $f = 0$.

Above result shows that direct summand of $T$-noncosingular is also $T$-noncosingular, the natural question here is that, what about direct sum of T-noncosingular?

The following example will answer our question and look at the condition that may generalised the answer to the problem.

**Example 5.2)** We have seen above that $\mathbb{Z}_p\ and\ \mathbb{Z}_{p^\infty}$ ( $\mathbb{Z}_{p^\infty}$ is divisible and divisible groups are T-noncosingular) are T-noncosingular. We will now show that their direct sum $G = \mathbb{Z}_p \oplus \mathbb{Z}_{p^\infty}$ is not T-noncosingular. Really, define $f$: G $\longrightarrow$ G by $f(k, c_n) = (0, kc_1)$. clearly $f$ is a homomorphism and $Imf\ = 0 \oplus < c_1 >$. since $< c_1 > \ll \mathbb{Z}_{p^\infty}, Imf \ll G$ and of course $f \neq 0$. So $G$ is not $T$-noncosingular.

The following proposition gives the condition for which direct sum of $T$-noncosingular abelian group to be $T$-noncosingular.

Proposition 5.1.7 [Tutuncu and Tribak 2009] Let $(H_i)_{i\,\in\,I}$ be a family be a family of subgroups of $G$, and $G = \oplus\ (H_i)_{i\,\in\,I}$ , then $G$ is T-noncosingular if and only if $H_i$ is $T$ - noncosingular related to $H_j$ for all $i, j \in I$.

Note that from the proposition above we can draw an important result as follow

**Corollary 5.1.8** Every semisimple group $C$ is $T$-noncosingular.

Proof; $C = \Sigma\mathbb{Z}_p = \ \oplus\ \mathbb{Z}_p$ where $p$ is prime, if $p \neq q$ then $Hom\big(\mathbb{Z}_p, \mathbb{Z}_q\big) = 0$, so $\mathbb{Z}_p$ is $T$-noncosingular related to $\mathbb{Z}_q$ by proposition 5.1.7 $C$ is $T$-noncosingular.

**Corollary 5.1.9** Every free group $F$ is $T$-noncosingular.

We know from [Rotman JJ 1982] $F = \sum_{x \in X} < x >$ where each $< x > \cong \mathbb{Z}$ and $X$ is a basis of $F$, therefore we can write $F = \sum \mathbb{Z}$. Then each $\mathbb{Z}$ is $T$-noncosingular related to itself and hence $F\ is\ T$-noncosingular.

**Corollary 5.1.10** Pure subgroup $H$ of a divisible group $D$ is $T$-noncosingular.

Proof: $n$H=H $\cap\, nD$, but since $D$ is divisible we can write $D = nD$ therefore $n$H=H $\cap\, nD$= H $\cap\, D = H$ since $H \leq G$ this means that $nH = H$ and that also mean $H$ is divisible, and divisible group is $T$-noncosingular .

**Proposition 5.1.11** Let $G = \bigoplus H_i$, $G$ is $T$-noncosingular if Rad $(H_i) = 0$ for each $i$.

Proof: follows from **proposition 5.1.3**

**Proposition 5.1.12** For a group $G$ with Rad $G \ll G$, the following are equivalent.

(1) If for every non zero $f$, $Imf \nsubseteq Rad\ G$.

(2) $G$ is $T$-noncosingular.

Proof:

(1) $\Rightarrow$ (2) I$mf \nsubseteq Rad\ G \ll G$ then $Imf$ is not small in $G$ for every $f$, means that $G$ is $T$-noncosingular.

(2) $\Rightarrow$ (1) $0 \neq f \in E$ =End (G) and assume that I$mf \subseteq Rad\ G$ then I$mf \subseteq Rad\ G \ll G$, we have I$mf \ll G$ by definition 5.02 it means $f = 0$. But $f \neq 0$ contradiction therefore I$mf \nsubseteq Rad\ G$.

**Theorem 5.1.13** A torsion group $G$ is T-noncosingular if and only if $G = D \oplus C$, where $D$ is divisible and $C$ is semi-simple and if $C$ has a direct summand isomorphic to $\mathbb{Z}_p$ for some prime $p$, then $D$ has no direct summand isomorphic to $\mathbb{Z}_{p^\infty}$(That is if $C_p \neq 0$ then $D_p = 0$).

Proof: ($\Longrightarrow$) Let $G$ be a torsion $T$-noncosingular group and $D$ be its maximal divisible subgroup. Then $G = D \oplus C$ for some $C \leq G$. Let $p$ be prime and $B$ be the basic subgroup of p – component $C_p$ of $C$. If $B$ has a direct summand $M$ isomorphic to $\mathbb{Z}_{p^n}$ with $n > 1$, then since $B$ is a pure subgroup of $C_p$, $M$ is a pure subgroup of $G$, hence $M$ is a direct summand of $G$ by corollary 3.5.7, therefore is $T$ - noncosingular. But we know that $\mathbb{Z}_{p^n}$ with $n > 1$, is not $T$-noncosingular. So $B$ is the direct sum of subgroups isomorphic to $\mathbb{Z}_p$, hence is semi-simple. Then $B$ is a bounded pure subgroup of $C_p$, therefore $C_p = B \oplus D_1$ for some divisible subgroup $D_1$ of $C_p$. But $C_p$ is reduced hence $D_1 = 0$ and also $C_p$ is semisimple. Then C $= \oplus C_p$ (where $p$ is prime) is also semisimple.

Now if $C \cong M \oplus N$ with $M \cong \mathbb{Z}_p$ and $D = K \oplus L$ with K $\cong \mathbb{Z}_{p^\infty}$, then there is a monomorphism $g: M \longrightarrow K$ with $Img \ll K$. Therefore for the endomorphism $f: G \longrightarrow G$ of $G = D \oplus$ C $= K \oplus L \oplus M \oplus N$ defined by $f(k, l, m, n) = (g(m), 0,0,0)$. We have $Imf = Img \oplus 0 \oplus 0 \oplus 0 \ll K \oplus L \oplus M \oplus N$ and $f \neq 0$ that is a contradiction with $T$-noncosingularity of $G$. So if C has a direct summand isomorphic to $\mathbb{Z}_p$ for some prime $p$, then $D$ has no direct summand isomorphic to $\mathbb{Z}_{p^\infty}$.

($\Leftarrow$) If the conditions are satisfied then $D$ is $T$-noncosingular related to $D$ and $C$ and also $C$ is $T$-noncosingular related to $D$ and $C$ therefore by proposition 5.1.7 $G = D \oplus C$ is $T$- noncosingular.

**Proposition 5.1.14** For a torsion group $G$, the following are equivalent

(1) $G$ is $T$-noncosingular and $Imf \nsubseteq R\text{ad}\, G$ for all non zero $f$

(2) $G$ is semi-simple.

Proof:

$(1) \Rightarrow (2)$ $G$ is torsion then by **theorem 3.1.2** we can write $G = \underset{p-prime}{\oplus G_p}$ and by **proposition 5.1.8** each $G_p$ is $T$-noncosingular related to $G_q$ for $p \neq q$ and with this condition and **proposition 5.1.12** Rad $G_p = 0$ for each $p$ then this means $pG_p = 0$, thus $G_p \cong \mathbb{Z}_p$, then $G$ is the direct sum of simple group and hence $G$ is semisimple.

$(2) \Rightarrow (1)$ $G$ is semisimple then by t**heorem 4.3.3** Rad $G$ = 0 and by **corollary 5.1.9** we can easily see that $G$ is $T$-noncosingular. Note that $Imf$ is not small in $G$, therefore $Imf \neq 0$ hence $Imf \nsubseteq R$ad $G$.

Theorem 5.1.13 fully characterized the condition for which of torsion group will be $T$-noncosingular and proposition 5.1.14 further supported the generalization. At this point we want to see whether or not the generalization of torsion-free group is also possible.

**Proposition 5.1.15** For a torsion-free group $G$, If $Rad^{k+1}\ G \trianglelefteq Rad^k\ G$ for $k = 0,1,\dots,n-1$, where $Rad^0\ G = G$ and $Rad^n\ G = 0$ then $G$ is $T$-noncosingular.

Proof: let $f\colon \mathrm{G} \longrightarrow \mathrm{G}$ be an endomorphism with $Imf \ll G$. Then $f(G) = Imf \leq Rad\ G$ and $f(Rad\ G) \leq Radf(G) \leq Rad(Rad\ G) = Rad^2G$. Similarly $f(Rad^2G) \leq Radf(Rad\ G) \leq Rad(Rad^2G) = Rad^3G$. Continuing in this way we will get $f(Rad^{n-1}G) \leq Rad^nG = 0$.

Since $Rad^{n-1}G \trianglelefteq Rad^{n-2}G$ for every $0 \neq a \in Rad^{n-2}G$, we have $0 \neq ma \in Rad^{n-1}G$, for some $m \in \mathbb{Z}$, therefore $mf(a) = f(ma) = 0$. since $G$ is torsion-free, $f(a) = 0$. So $f(Rad^{n-2}G) = 0$. Continuing tin this way we get $f(G) = 0$, that is $f = 0$ , hence $G$ is $T$-noncosingular.

**Example 5.3** Let $B = \{\frac{a}{p_1^2 p_2^2 \ldots\ldots\ldots} \mid a \in \mathbb{Z},\ p_i \; is\; prime\; for\; each\; i\}$, we want check whether $B$ is $T$ - noncosingular or not.

Proof: remember that A= $\left\{\frac{n}{m} \in Q : m \; is\; a\; square\; free, m = p_{1,}p_2, \ldots\ldots\ldots, p_k\right\}$, therefore $Rad\, B = A \; and \; Rad\, A = \mathbb{Z} \; while\; Rad\, \mathbb{Z} = 0$ therefore $Rad^3\, B = 0$ and $\mathbb{Z} = Rad^2\, B \trianglelefteq Rad^1 B = A$ , therefore $B$ satisfy all the condition of **proposition 5.1.13** hence $B$ is $T$ -noncosingular.

The natural question that may arise here is that, what if the Rad $G$ of a torsion-free group $G$ is not essential in $G$ and or the $Rad^n\, G \neq 0$. The following example will suggest something important for us.

**Example 5.4)** Let $B_p = \{\frac{a}{m} . \mid (m,p) = 1\}$, then $B_p$ is not $T$ -noncosingular.

Proof: Note that $B_p$is torsion-free group and $pB_p$ is the largest subgroup of $B_p$ but $pB_p \ll B_p$. Now take a non zero endomorphism $f : B_p \longrightarrow B_p$ defined by $f(x) = px$, then $Imf \leq pB_p \ll B_p$. Therefore $Imf \ll B_p$ and hence $B_p$ is not $T$ -noncosingular.

# CHAPTER SIX

## CONCLUSIONS

For a torsion group, we are able to fully characterized the notion of $T$-noncosingular abelian group and precisely stated that, for a torsion group to be $T$-noncosingular the group $G$, must be decomposed as $G = D \oplus C$, where $D$ is divisible and $C$ is semi-simple. Also we established that, for a torsion $T$-noncosingular group $G$ with $Imf \nsubseteq Rad\, G$ for all nonzero $f$ produces semi-simple group $G$.

Our characterization for torsion group is surprisingly working; but the situation of torsion-free group is still subject to further research; however we provide a result that gives more information of the notion for every abelian group $G$.

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