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DIFFERENT SEMANTICS OF MODAL LOGIC

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Yüksek lisans tezi olarak sunduğum ‘Different Semantics of Modal Logic’ adlı çalışmanın tarafımdan bilimsel ahlak ve geleneklere aykırı düşecek bir yardıma başvurmaksızın yazıldığını ve yararlandığım eserlerin referanslarda gösterilenlerden oluştuğunu, bunlara atıf yapılarak yararlanılmış olduğunu belirtir ve bunu onurumla doğrularım.

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ÖZET

MODAL LOJİĞİN FARKLI SEMANTİKLERİ

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Yüksek Lisans Tezi, Matematik Bölümü

Tez Danışmanı : Prof. Dr. Mehmet TERZİLER

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Bu tez 3 bölümden oluşmaktadır. Birinci bölümde, bağıntısal veya temel modal lojiğin Kripke semantiği incelenmiş S4 gibi iyi tanınan ve temel lojiğin sağlamlık ve tamlık teoremleri kanıtlanmıştır.

İkinci bölümde, topolojik semantik ve bazı lojiklerin tanımlanabilirliği incelenmiştir. İki temel modal operatörümüz ‘kutu’ ve ‘elmas’ sırasıyla, topolojik iç ve kapanış olarak yorumlanmaktadır. İyi bilinen McKinsey-Tarski sonucunun S4 ün topolojik semantiğe göre tam olduğu modern bir yaklaşım yoluyla kanıtlanmıştır. Bu semantikle birlikte, bazı topolojik uzaylar tanımlanmıştır.

Üçüncü ve son bölümde, diğer tüm modal semantikleri genellediğimiz komşuluk semantiğini ele alıyoruz. Kripke veya Topolojik semantik açısından geçerli bilinen her şey bu semantiğe göre geçerli olmayabildiği gösterilmiştir. Bu semantiğin genel bir semantik olduğunu gösteren örnekler ve sonuçlar verilmiştir.

Anahtar Kelimeler : Modal Lojik, sağlamlık, tamlık, Kripke Semantiği, Bağıntısal Modeller, Topolojik Semantik, Topolojik Modeller, Komşuluk Semantiği

ABSTRACT

DIFFERENT SEMANTICS OF MODAL LOGIC

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This thesis consists of three chapters. In the first chapter, relational or Kripke semantics of the basic modal logic is studied. For well-known and basic logics such as S4 the soundness and completeness theorems are proved.

In the second chapter, topological semantics and definability of certain logics are analysed. Our two basic modal operators ‘box’ and ‘diamond’ are interpreted as the topological interior and closure, respectively. The well-known result of McKinsey-Tarski that S4 is complete with respect to topological semantics is proved via modern approach. With this semantics, certain topological spaces are defined.

In the third and last chapter, we deal with the neighborhood semantics that generalizes all the other modal semantics. All that is known valid with respect to Kripke or Topological semantics may not be so under this semantics. We give examples and results showing that this is indeed a general semantics.

Key Words : Modal Logic, soundness, completeness, Kripke Semantics, Relational Models, Topological Semantics, Topological Models, Neighborhood Semantics.

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Introduction

In classical propositional and predicate logic, every formula is either true or false in any model; that is, the classical logic is a two-valued logic. But there are situations where we need to distinguish between different modes of truth, such as ‘*necessarily true*’, ‘*known to be true*’, ‘*believed to be true*’ and ‘*always true in the futur*’ (with respect to time). For example, the sentence ‘*The Eiffel Tower is located on Champ de Mars in Paris*’ expresses something that is true today, but was false some years ago. Moreover, it might be false again some time in futur. On the other hand, the mode of truth of the sentence

‘*The Earth revolves around the Sun*’ is more stable with respect to time, since it expresses something that is true and maybe will be true for ever in the future, but it is not necessarily true in the sense that the Earth (or the sun) might transform into a black hole.

However, most people would agree that the statement ‘*The area of a square is equal to the product of its two sides*’ expresses something that is both necessarily true as well as always true. But it does not enjoy all modes of truth, for instance it may not be believed to be true (for example, by someone who is mistaken) or known to be true (for example, by someone that hasn’t learned mathematics).

There are also practical examples where reasoning about different modes of truth is helpful. For example, each agent of a multi-agent system in computer science may have different knowledge about the system. In such situation, a sentence is ‘*necessarily*’ true when known. Of course, not every sentence needs to be necessarily true in this sense.

In fact, we use the same way of reasoning in all modes of truth. A sentence ϕ , if true will be so with respect to the current state of affairs, i.e. how the world actually is, but (depending on ϕ) we might be able to conceive a different world where ϕ is false, and if this is the case ϕ will not be necessarily true.

In this thesis, we will develop a general framework in which we will be able to reason about situations as the ones above. We will concentrate especially on three semantics : Kripke Semantics, Topological Semantics, and Neighborhood Semantics.

Historically, the standard semantics of modal logics were given by Saul Kripke in the 1960s in the form that is now known as '*Kripke Semantics*'. These semantics, while very successful for investigating propositional modal systems, fail to accomodate first-order semantics; in particular, consistent extensions of many first-order model systems fail to be complete on any Kripke-style models [3].

The Project of relating topology to modal logic begins with work of Alfred Tarski and J.C.C. McKinsey [8].

The basic idea is to study the laws of the '*interior*' operation on subsets of a topological space and its dual, the '*closure*' operation. Interpreting the modal operator \Box as interior and the modal operator \Diamond as closure , modal logic and topological spaces are connected.

We will see that definitions of an open set and a closed set will coincide with the semantics of the unary operators \Box , box, and \Diamond , diamond. Using this semantics, the definability of certain properties will be given as examples.

Neighborhood semantics is a generalization of the Kripke, or relational, semantics for modal logic invented by Dana Scott and Richard Montague, independently [9] and [12]. In this semantics, a function replaces the relation of Kripke frame, and many valid formulas with respect to Kripke semantics as well as topological semantics are no longer true.

For more knowledge about these concepts, we give some basic books such as

1. [2], in this book, Chapters 7-9 cover basic results about classical modal logic, and in particular, neighborhood semantics is studied in great details.
2. [1], the first four chapters of this book contain main results and historical notes of the evolution of modal logic.

Chapter 1

KRIPKE SEMANTICS

The language of the thesis will be the language of the propositional logic augmented by two binary modal operations, box and diamond. Before introducing Kripke semantics, we describe Basic Modal Logic.

Basic Modal Logic

A logic is studied by both syntax and semantics. These two approaches are shown to be equivalent by means of Correctness (Soundness) and Completeness Theorems.

1.1 Syntax

The language of basic modal logic is an extension of classical logic. We add two unary connectives (operators) \Box and \Diamond to the language of the propositional logic. Let At denote the set of propositional letters p, q, r, \dots also called atomic formulas or atoms.

$$\varphi ::= \perp \mid \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \mid \Box\varphi \mid \Diamond\varphi$$

This means that formulas are generated by Boolean connectives and new operators \Box and \Diamond .

This means that formulas are generated by Boolean connectives and new operators \Box and \Diamond .

Remark 1.1.2 Typically only one of \Box and \Diamond is taken as primitive and the other is ‘defined’ to be the dual, for example, $\Diamond\varphi$ is sometimes defined to be $\neg\Box\neg\varphi$, and also $\Box\varphi$ to be $\neg\Diamond\neg\varphi$. We have opted to take both \Box and \Diamond as primitive.

$(q \wedge \neg\Diamond p)$ and $\Box p$ are formulas, while $p\Box\neg\rightarrow p$ and $\vee p\Diamond q$ are not. Just as in predicate logic, the unary connectives \Box and \Diamond bind most closely so that $\Box p \vee q$ is read as $(\Box p) \vee q$ and not $\Box(p \vee q)$.

The new connectives \Box and \Diamond are read ‘*box*’ and ‘*diamond*’ respectively, and are dual each other similarly to \forall and \exists in predicate logic. And just as \forall and \exists , are read as ‘*for every*’ and ‘*there exists*’ respectively, we will see that *box* and *diamond* possess similar readings. Although the readings will be different depending on the situation, initially \Box is read as ‘*necessarily*’ and \Diamond as ‘*possibly*’. In such a logic, there are some formulas we might regard as being correct principles, for example, $\Box\phi \rightarrow \Diamond\phi$ ‘*whatever is necessary is possible*’ or $\phi \rightarrow \Diamond\phi$ ‘*whatever is, is possible*’. However, other formulas may be harder to decide. For example, should $\phi \rightarrow \Box\Diamond\phi$ ‘*whatever is, is necessarily possible*’ be regarded as a general truth about necessity and possibility? We need a precise semantics to bring clarity to questions like these.

Remark 1.1.3 To generate the set of formulas in propositional logic, usually we take one of the $\{\neg, \vee\}$, or $\{\neg, \wedge\}$, or $\{\neg, \rightarrow\}$ here we could as well have defined a formula by only $\neg, \wedge, \rightarrow, \Box$.

1.2 Semantics

We give some mathematical content to suggestive discussion above. A model in propositional logic is simply a valuation function assigning truth values to the set of atoms, i.e. a function $v : At \rightarrow \{\top, \perp\}$.

We now consider models in which an atom can have different truth values at different states.

Definition 1.2.1 A Kripke frame is a pair $\langle W, R \rangle$ where W is a nonempty set and R is a binary relation on W .

A Kripke frame is also known as

- directed graph (graph theory)
- relational structure (modal logic)
- network (network theory)
- labelled transition system (computer science)
- automaton (computer science)

Two basic logics are first order logic and modal logic for graph theory.

First order logic describes a graph using quantifiers and relations. It can provide a complete description of a finite graph. However, the truths of the logic are generally undecidable in the sense that the answer to the question ‘*is ϕ true in some model/graph?*’ cannot necessarily be answered in a finite number of time-steps.

Modal logic describes properties of vertices in a graph using locally defined quantifiers, namely box and diamond. It can only express bisimulation equivalence classes of a graph (highly relevant in modelling computation). The logic is generally decidable.

Definiton 1.2.2 A model , M , in basic modal logic is a triple $\langle W, R, V \rangle$, where

- W is a set of states or worlds,

- R is a relation $R \subseteq W \times W$,

- And $V: At \rightarrow P(W)$ is a function, a valuation, assigning a subset of W to each atom, where $P(W)$ denotes the power set of W

[Sometimes a model M is defined by means of a labelling function, $L: W \rightarrow P(At)$, where $P(At)$ represents the powerset of At .]

These models are called Kripke models after Saul Kripke who was the first to introduce them in the 1950s. Intuitively, $w \in W$ is a possible world and R is an accessibility relation between worlds. That is, $w R w'$ (which is used for $(w, w') \in R$) means that w' is accessible from w . This intuition will be made precise below.

Relational structures have advantage to be represented graphically. This is the case here, let us give some examples.

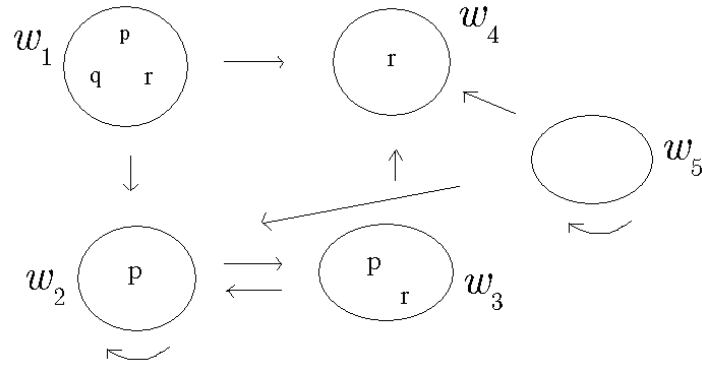
Example 1.2.3 Consider the Kripke model $M = \langle W, R, V \rangle$, where

$$W = \{w_1, w_2, w_3, w_4, w_5\}$$

$$R = \{(w_1, w_2), (w_1, w_4), (w_2, w_2), (w_2, w_3), (w_3, w_2), (w_3, w_4), (w_5, w_5), (w_5, w_2), (w_5, w_4)\}$$

, and $V(p) = \{w_1, w_2, w_3\}$, $V(q) = \{w_1\}$, $V(r) = \{w_1, w_3, w_4\}$

Then we can picture M as follows



(Fig.1.1)

where an arrow $w_i \rightarrow w_j$ means that $w_i R w_j$ and $w_i \rightarrow w_i$ means $w_i R w_i$, Note that we could take L instead of V by defining :

$$L(w_i) = \begin{cases} \{p, q, r\} & i = 1 \\ \{p\} & i = 2 \\ \{p, r\} & i = 3 \\ \{r\} & i = 4 \\ \emptyset & i = 5 \end{cases}$$

Example 1.2.4 A Kripke modal $M = \langle W, R, L \rangle$ can be used to describe how truth values vary over time. A common example is when $W = \mathbb{N}$ and R is the usual ordering \leq of the natural numbers.

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow \dots$$

(Fig.1.2)

Then we can think of W as a set of points in time and R as the relation of being ahead in time. Then $L(t)$ will describe the truth values of propositions at time $t \in W$.

Remark 1.2.5 If we remove R from Kripke model, that is, if we consider the model $\langle \{*\}, \emptyset, V \rangle$ then we see that it is just an ordinary model for propositional logic :

$$V : At \rightarrow \{\top, \perp\} ,$$

$$V(p) = \begin{cases} \top, & \text{if } p \in L(*) \\ \perp, & \text{if } p \notin L(*) \end{cases}$$

Definition 1.2.6 Let $M = \langle W, R, V \rangle$ be a model in basic modal logic. Suppose $w \in W$ and φ is a formula. We will define when φ is ‘true in the world’ w . This is done via a satisfaction relation $w \models \varphi$ by structural induction on φ :

- $w \models \top$
- $w \not\models \perp$
- $w \models p$ (iff) $p \in V(w)$
- $w \models \neg \varphi$ (iff) $w \not\models \varphi$
- $w \models \varphi \wedge \psi$ (iff) $w \models \varphi$ and $w \models \psi$
- $w \models \varphi \vee \psi$ (iff) $w \models \varphi$ or $w \models \psi$
- $w \models \varphi \rightarrow \psi$ (iff) $w \models \psi$ whenever $w \models \varphi$, or equivalently
(iff) $w \not\models \varphi$ or $w \models \psi$
- $w \models \varphi \leftrightarrow \psi$ (iff) $w \models \varphi$ (iff) $w \models \psi$
- $w \models \Box \varphi$ (iff) for each $u \in W$ with wRu we have $u \models \varphi$
- $w \models \Diamond \varphi$ (iff) there exists $u \in W$ such that wRu and $u \models \varphi$

When $w \models \varphi$, we say that ‘ w satisfies / forces φ ’ or ‘ φ is true in world w /at state w ’.

Comment: The first eight clauses are clear from propositional logic, the only difference is that an atom p can be true at many worlds w . The interesting cases are the interpretations of box and diamond. For $\Box \phi$ to be true at w , ϕ must be true at every world u accessible from w , and for $\Diamond \phi$ to be true at w there must be at least one world u related to w such that ϕ is true at u . It is to be noted that \Box and \Diamond act a bit like the quantifiers \forall and \exists , but \Box and \Diamond act over states. This interpretation is usually called ‘*possible worlds semantics*’.

Example 1.2.7 Consider the model of example 1.2.3. Then according to definition 1.2.6, $w_2 \models \Box p$ and $w_1 \models \Diamond (r \wedge \Box q)$. Indeed, since $w_2 R u$ implies that $u = w_2$ or $u = w_3$ (see fig 1.1), we have $w_2 \models p$ and $w_3 \models p$, i.e., $w_2 \in V(p)$ and $w_3 \in V(p)$ (or equivalently, $p \in L(w_2)$ and $p \in L(w_3)$). Now since $w_1 R w_4$ and $w_4 \models r$ and $w_4 \not\models \Box q$ because there is no $u \in W$ such that $w_4 R u$. On the other hand, $w_5 \not\models \Box p$ and $w_5 \models \Diamond r$.

Definition 1.2.8 A model $M = \langle W, R, V \rangle$ is said to satisfy a formula ϕ if every state in the model satisfies ϕ . Thus, we write $M \models \phi$ if and only if $w \models \phi$ for every $w \in W$.

For example, if we again consider the model in Example 1.2.3, we see that $w_1, w_3, w_4 \models r$ and $w_2, w_5 \models \Diamond p$. Hence the modal formula $r \vee \Diamond p$ is satisfied by M , i.e., $M \models r \vee \Diamond p$.

We now define modal semantic entailment, similar to logical consequence in classical logic.

Definition 1.2.9 Let Γ be a set of formulas. Then we say that Γ ‘*semantically entails*’ a formula ϕ if for any world w in any model $M = \langle W, R, V \rangle$ we have

$w \models \phi$ whenever $w \models \psi$ for every $\psi \in \Gamma$. In this case, we write $\Gamma \models \phi$.

Two formulas ϕ and ψ are said to be semantically equivalent when they semantically entail each other, and then we write $\phi \equiv \psi$.

Proposition 1.2.10 We have the following equivalences :

- (a) $\Box \phi \equiv \neg \Diamond \neg \phi$ and $\Diamond \phi \equiv \neg \Box \neg \phi$
- (b) $\Box (\phi \wedge \psi) \equiv (\Box \phi \wedge \Box \psi)$
- (c) $\Diamond (\phi \vee \psi) \equiv (\Diamond \phi \vee \Diamond \psi)$

Proof Let $M = \langle W, R, V \rangle$ be an arbitrary model.

- (a) Let $w \in W$ be any world. Suppose $w \models \Box \varphi$, then $u \models \varphi$ for every $u \in W$ such that wRu . So there cannot be a world $u \in W$ such that wRu and $u \models \neg \varphi$, but then $w \not\models \Diamond \neg \varphi$. Hence $w \models \neg \Diamond \neg \varphi$. > Thus $\Box \varphi \models \neg \Diamond \neg \varphi$. Conversely, if $w \models \neg \Diamond \neg \varphi$, then $w \not\models \Diamond \neg \varphi$ so that there is no world u such that wRu and $u \models \neg \varphi$. Hence, for any $u \in W$ such that wRu , we must have $u \models \varphi$. But then $w \models \Box \varphi$. Thus $\neg \Diamond \neg \varphi \models \Box \varphi$, hence $\Box \varphi \equiv \neg \Diamond \neg \varphi$. The second equivalence follows easily from the first one. \otimes
- (b) Let $w \in W$ be any world. Suppose $w \models \Box(\varphi \wedge \psi)$, then for every u such that wRu , $u \models \varphi$ and $u \models \psi$ by Definition 1.2.6. But then of course $w \models \Box \varphi$ and $w \models \Box \psi$, i.e., $w \models \Box \varphi \wedge \Box \psi$. Thus $\Box(\varphi \wedge \psi) \models (\Box \varphi \wedge \Box \psi)$. Conversely, if $w \models \Box \varphi \wedge \Box \psi$, then by Definition 1.2.6 $w \models \Box \varphi$ and $w \models \Box \psi$, i.e. for every u such that wRu , $u \models \varphi$ and $u \models \psi$, that is, $u \models \varphi \wedge \psi$. As u is arbitrary, we have $w \models \Box(\varphi \wedge \psi)$, hence $\Box \varphi \wedge \Box \psi \models \Box(\varphi \wedge \psi)$. Therefore we obtain $\Box(\varphi \wedge \psi) \equiv (\Box \varphi \wedge \Box \psi)$. \otimes
- (c) Let $w \in W$ be any world. Suppose $w \models \Diamond(\varphi \vee \psi)$, then by Definition 1.2.6, there exists u such that wRu and $u \models \varphi \vee \psi$. Hence $u \models \varphi$ or $u \models \psi$. Thus $w \models \Diamond \varphi$ or $w \models \Diamond \psi$. So $\Diamond(\varphi \vee \psi) \models \Diamond \varphi \vee \Diamond \psi$. \otimes

Conversely, suppose $w \models \Diamond \varphi \vee \Diamond \psi$, then by definition 1.2.6, $w \models \Diamond \varphi$ or $w \models \Diamond \psi$. Hence there exists $u \in W$ such that wRu and $u \models \varphi \vee \psi$, so $w \models \Diamond(\varphi \vee \psi)$. Thus $\Diamond \varphi \vee \Diamond \psi \models \Diamond(\varphi \vee \psi)$, and we get the required equivalence. \otimes

Remark 1.2.11 The equivalences in proposition 1.2.10 are very ‘*natural*’ in the sense that the quantifiers \forall and \exists distribute over \wedge and \vee , respectively. We will see that \Box will also distribute over \rightarrow , which is a fundamental property for modal logic. Also, we know that $\forall \equiv \neg \exists \neg$ and $\exists \equiv \neg \forall \neg$.

Example 1.2.12 We have that in general

$$\Box (\varphi \vee \psi) \not\equiv (\Box \varphi \vee \Box \psi)$$

$$\Diamond (\varphi \wedge \psi) \not\equiv (\Diamond \varphi \wedge \Diamond \psi).$$

To see that $\Box (\varphi \vee \psi) \not\equiv (\Box \varphi \vee \Box \psi)$, it suffices to find a model in which one formula holds while the other fails. For the first, let $W = \{w_1, w_2, w_3\}$, $R = \{(w_1, w_2), (w_1, w_3)\}$, and $V(p) = \{w_3\}$, $V(q) = \{w_2\}$

Then we have $w_1 \Vdash \Box (p \vee q)$, since $w_1 R w_2$ and $w_1 R w_3$ and $w_2 \Vdash p \vee q$ and $w_3 \Vdash p \vee q$. However, $w_1 \not\Vdash \Box p \vee \Box q$ since $w_1 R w_2$ and $w_2 \not\Vdash p$ and $w_3 \not\Vdash q$. Thus they are not equivalent. (See fig 1.3)

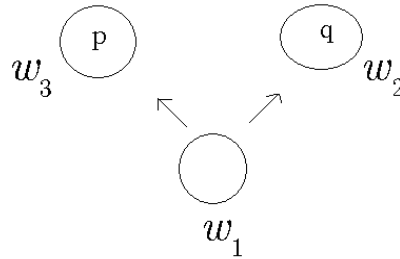


Fig.1.3

It can be shown that the same model works for the second inequivalence.

Now, we need a notion of validity. We know valid formulas, also called tautologies, from classical logic. Here a valid formula will be true with respect to every valuation as well as underlying frame $\langle W, R \rangle$.

Definition 1.2.13 We say that a formula φ is ‘*valid*’ if it is true in every world of every model. We denote this by $\Vdash \varphi$.

For example, formulas of the above proposition 1.2.10 are valid formulas. Another important valid formula, which we shall prove, is the following :

$$(K) \quad \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$$

This formula is called K (honoring S. Kripke).

Proposition 1.2.14 K is valid.

Proof Let $M = \langle W, R, V \rangle$ be any model and let $w \in W$. Assume that $w \models \Box(\varphi \rightarrow \psi)$ and $w \models \Box\varphi$. This holds if and only if for every $u \in W$ such that wRu , we have $u \models \varphi \rightarrow \psi$ and $u \models \varphi$, which implies that $u \models \psi$ for every u such that wRu . But this on the other hand holds if and only if $w \models \Box\psi$. Hence $w \models \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$. As M was arbitrary, we have $\models K$. \otimes

Many other formulas, such as $\Box\varphi \rightarrow \varphi$, $\Box\varphi \rightarrow \Box\Box\varphi$, $\Diamond\top$, $\varphi \rightarrow \Box\Diamond\varphi$, are not valid. So for each situation, or reading of \Box , and its dual \Diamond , we will restrict the classes of models so that the desired formulas are valid with respect to this class.

Notice that \Box and \Diamond are interpreted by means of R . So the question is what properties the relation R should have in the various cases. This leads us to consider relations between first order logic and modal logic.

1.3 Correspondence

We will see that some elementary classes of models correspond to simple formulas in basic modal logic. This will yield a connection between what formulas should be valid and what general structure the models should have in each situation.

1.3.1 Frame Correspondence

Definition 1.3.1 A structure $\langle W, R \rangle$ with W a nonempty set and R a binary relation on W is called a ‘*frame*’ and is denoted F .

Thus a frame F is the underlying structure of any model M , and so from any model we can extract a frame by simply forgetting about the valuation function V .

Definition 1.3.2 A formula φ is ‘valid’ on a frame F , written $F \models \varphi$, if for every valuation function V and each $w \in W$, we have $M, w \models \varphi$ where

$$M = \langle W, R, V \rangle.$$

Recall that we defined validity of a formula φ , $\models \varphi$, by saying that φ is true at every point of every model, but we could equivalently say that a formula φ is valid when $F \models \varphi$ for all frames F .

Proposition 1.3.3 Let $F = \langle W, R \rangle$ be a frame. Then

- (1) R is reflexive if and only if $F \models \Box\varphi \rightarrow \varphi$
- (2) R is transitive if and only if $F \models \Box\varphi \rightarrow \Box\Box\varphi$.

Proof

(1) **CN (\Rightarrow)**: Suppose that R is reflexive and let V be a valuation function so that we get a model $M = \langle W, R, V \rangle$. We want to show that $M \models \Box\varphi \rightarrow \varphi$, so let $w \in W$ such that $w \models \Box\varphi$. Then for all $u \in W$ such that wRu , we have $u \models \varphi$. Since R is reflexive, wRw , hence $w \models \varphi$. But then we have $w \models \Box\varphi \rightarrow \varphi$, and $F \models \Box\varphi \rightarrow \varphi$ since w was arbitrary.

CS (\Leftarrow): Suppose $F \models \Box\varphi \rightarrow \varphi$. In particular, we then have $F \models \Box p \rightarrow p$. Now let $w \in W$ such that $w \notin V(p)$ and $u \in V(p)$ for each $u \in V$ such that wRu . Assume that we don’t have wRw . Then $w \models \Box p$. But then since F satisfies $\Box p \rightarrow p$, we also must have $w \models p$. But this contradicts the assumption that $w \notin V(p)$. Hence it must be the case that wRw . Since w was arbitrary, this shows that R is reflexive. \otimes

(2) **CN (\Rightarrow)**: Suppose that R is transitive and let $M = \langle W, R, V \rangle$ be a model. We want to show that $M \models \Box\varphi \rightarrow \Box\Box\varphi$. Let $w \in W$ such that $w \models \Box\varphi$. We then need to see that for every $u \in W$ such that wRu and every $v \in W$ such that uRv , we have $v \models \varphi$ i.e. $w \models \Box\Box\varphi$. But if wRu and uRv , then wRv since R is transitive, and together with $w \models \Box\varphi$ we then have $v \models \varphi$. Hence $w \models \Box\Box\varphi$. This shows that $F \models \Box\varphi \rightarrow \Box\Box\varphi$.

CS(\Leftarrow): Suppose $F \models \Box\phi \rightarrow \Box\Box\phi$. In particular, we have $F \models \Box p \rightarrow \Box\Box p$. Let $w, u, v \in W$ be such that wRu and uRv . We want to show that wRv . Let V be a valuation function such that $v \notin V(p)$ but $w \in V(p)$ for all other w . Assume, to the contrary, that we don't have wRv . Then $w \models \Box\phi$ and hence $w \models \Box\Box\phi$ since by hypothesis $F \models \Box\phi \rightarrow \Box\Box\phi$. But then $u \models \Box\phi$, since wRu , and $v \models p$, since uRv , which contradicts our assumption that $v \notin V(p)$. Hence, we must have wRv . This shows that R is transitive. \otimes

We give below some well-known and useful correspondences.

T : Frame validity of $\Box\phi \rightarrow \phi$ corresponds to reflexivity of R

B : Frame validity of $\phi \rightarrow \Box\Diamond\phi$ corresponds to symmetry of R

D : Frame validity of $\Box\phi \rightarrow \Diamond\phi$ corresponds to R being serial

4 : Frame validity of $\Box\phi \rightarrow \Box\Box\phi$ corresponds to transitivity of R

5 : Frame validity of $\Diamond\phi \rightarrow \Box\Diamond\phi$ corresponds to R being Euclidean

For more correspondences [10]

We will prove the last correspondence, but before we give a definition.

Definition 1.3.4 We say that a basic modal logic formula ϕ 'defines' a property P of a frame $F = \langle W, R \rangle$, if $F \models \phi$ if and only if R has the property P .

For example, one can show that $\Diamond\top$ and $D:\Box\phi \rightarrow \Diamond\phi$ define the same property; this corresponds to seriality of the accessible relation: $\forall x\exists y xRy$.

Proposition 1.3.5 R is Euclidean if and only if $F \models \Diamond\phi \rightarrow \Box\Diamond\phi$.

Proof A relation R is Euclidean if for every $w, u, v \in W$, wRu and wRv implies that uRv .

CN (\Rightarrow): Suppose that R is Euclidean. Let $M = \langle W, R, V \rangle$ be any model and $w \models \Diamond\phi$. Then there is $v \in W$ such that wRv and $v \models \phi$. Now suppose $u \in W$ with wRu . Then uRv since R is Euclidean. But then we have $u \models \Diamond\phi$, and hence $w \models \Box\Diamond\phi$, i.e., $w \models \Diamond\phi \rightarrow \Box\Diamond\phi$.

CS (\Leftarrow) : Assume that F is non-Euclidean. Then there must be $w, u, v \in W$ such that wRu, wRv but not uRv . We will try to falsify $\diamond \phi \rightarrow \Box \diamond \phi$ by finding a valuation function V such that $w \models \diamond \phi$ and $w \not\models \Box \diamond \phi$. That is, if we consider $\diamond p \rightarrow \Box \diamond p$, we have to make p true at some R -successor of w and false at all R -successors of some R -successor of w . Let V be given by $x \in V(p)$ (iff) it is not the case that uRx .

$$\text{Then } v \in V(p) \text{ while } \{x | uRx\} \cap \{x | x \in V(p)\} = \emptyset.$$

Now clearly $u \not\models \diamond p$, so that $w \not\models \Box \diamond p$. On the other hand, since we have $v \models p$ and wRv , we get $w \models \diamond p$. Hence, $F \not\models \diamond \phi \rightarrow \Box \diamond \phi$. \otimes

1.4 Normal Modal Logics

Given a class of frames \mathbf{F} , we denote by $L_{\mathbf{F}}$ the set of formulas valid on every frame in \mathbf{F} . For example, if \mathbf{F} is the class of reflexive frames, we know that $\Box \phi \rightarrow \phi \in L_{\mathbf{F}}$, if it is the class of symmetric frames $\phi \rightarrow \Box \diamond \phi \in L_{\mathbf{F}}$, etc. We can ask whether there are syntactic mechanisms capable of generating $L_{\mathbf{F}}$. And are such mechanisms able to cope with the associated semantic consequence relation.

We are going to define a Hilbert - style axiom system, called \mathbf{K} , which is a ‘*minimal*’ system for reasoning about frames.

Definition 1.4.1

A \mathbf{K} – ‘*proof*’ is a finite sequence of formulas, each of which is an ‘*axiom*’, or follows from one or more earlier items in the sequence by applying a ‘*rule of proof*’. The axioms of \mathbf{K} are all instances of propositional tautologies and :

$$\mathbf{K} : \Box (\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$$

$$\text{Dual} : \diamond \phi \leftrightarrow \neg \Box \neg \phi$$

The rules of proof of \mathbf{K} are :

‘*Modus ponens*’ : given ϕ and $\phi \rightarrow \psi$, prove ψ .

‘*Uniforme substitution*’ : given ϕ , prove ψ , where ψ is obtained from ϕ by replacing proposition letters or atoms in ϕ by arbitrary formulas.

‘Rule of necessitation’ : given φ , prove $\Box \varphi$.

A formula φ is **K**-‘provable’ if it occurs as the last item of some **K**- proof , in this case we write $\vdash_{\mathbf{K}} \varphi$.

Remark 1.4.2

- (i) Modus ponens preserves validity ; that is , if $\vDash \varphi$ and $\vDash \varphi \rightarrow \psi$, then also $\vDash \psi$. So it is a correct rule for reasoning about frames. Furthermore , modus ponens preserves ‘global truth’ (if $M \vDash \varphi$ and $M \vDash \varphi \rightarrow \psi$, then $M \vDash \psi$) and ‘satisfiability’ (if $M, w \vDash \varphi$ and $M, w \vDash \varphi \rightarrow \psi$, then $M, w \vDash \psi$). Thus, modus ponens is also a correct rule for reasoning about models.
- (ii) Uniform substitution mirrors the fact that validity has nothing to do with particular assignments , i.e. if a formula is valid, this does not depend on the particular values of its atoms. Uniform substitution preserves validity, but it does not preserve neither global truth nor satisfiability.
- (iii) The rule of necessitation might look some what odd, since clearly $\varphi \rightarrow \Box \varphi$ is not valid. However, this rule preserves validity and global truth.
- (iv) The axiom **K** is sometimes called the ‘distribution axiom’, and as we saw earlier, it is a valid formula.
- (v) The reason for having the Dual axiom is that we did not define \Diamond using box ; it is also valid.
- (vi) **K** is the minimal modal Hilbert system in the following sense : All its axioms are valid and all its rules of inference preserve validity, hence all **K**-provable formulas are valid. This leads us to the concepts of soundness and completeness. Before, we give an example of **K**-provability.

Example 1.4.3

The formula $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ is valid on any frame, so it should be **K**-provable. Indeed,

- 1) $\vdash p \rightarrow (q \rightarrow p \wedge q)$ (Propositional tautology)
- 2) $\vdash \Box(p \rightarrow (q \rightarrow p \wedge q))$ 1, Necessitation rule
- 3) $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ K axiom
- 4) $\vdash \Box(p \rightarrow (q \rightarrow p \wedge q)) \rightarrow (\Box p \rightarrow \Box(q \rightarrow p \wedge q))$ 3, Uniform Substitution
- 5) $\vdash \Box p \rightarrow \Box(q \rightarrow p \wedge q)$ 2,4 Modus Ponens
- 6) $\vdash \Box(q \rightarrow p \wedge q) \rightarrow (\Box q \rightarrow \Box(p \wedge q))$ 3, Uniform Substitution
- 7) $\vdash \Box p \rightarrow (\Box q \rightarrow \Box(p \wedge q))$ 5,6 Propositional Syllogism
- 8) $\vdash (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ 7, Propositional logic.

The system **K** is too weak to validate many formulas ; for example, if we are interested in validity only on transitive frames, the system **K** is not able to show that the formula $\Box \phi \rightarrow \Box \Box \phi$ is valid, hence is not **K**-provable. For this reason we will introduce the following concept.

Definition 1.4.4

A ‘normal modal logic’ L is a set of formulas of basic modal logic, with the following properties :

• L contains all propositional tautologies

• L contains all instances of **K** :

$$\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$$

• L contains all instances of the Dual :

$$\Diamond \phi \leftrightarrow \neg \Box \neg \phi$$

• L is closed under modus ponens, uniform substitution, and the rule of necessitation.

Some well-known normal systems are as follows :

$$\begin{aligned} \mathbf{T} &= \mathbf{K} + \mathbf{T} ; & \mathbf{T} : \Box \varphi \rightarrow \varphi & \text{ (reflexivity)} \\ \mathbf{S4} &= \mathbf{K} + \mathbf{T} + 4 ; & 4 : \Box \varphi \rightarrow \Box \Box \varphi & \text{ (transitivity)} \\ \mathbf{S5} &= \mathbf{K} + \mathbf{T} + 4 + 5 ; & 5 : \Diamond \Box \varphi \rightarrow \Box \varphi & \text{ (Euclidean)} \\ \mathbf{KD45} &= \mathbf{K} + \mathbf{D} + 4 + 5 ; & \mathbf{D} : \Box \varphi \rightarrow \Diamond \varphi & \text{ (seriality)} \end{aligned}$$

Among these systems, S4 is of particular importance in the content of topological spaces.

1.5 Soundness and Completeness

Soundness and completeness are key requirements of any logic. A logical system has the soundness property if and only if its rules of inference prove only formulas that are valid with respect to its semantics. Soundness is among the most fundamental properties of mathematical logic ; it provides the initial reason for counting a logical system as desirable. The completeness property means that every validity (truth) is provable. Together they imply that all and only validities are provable.

Most proofs of soundness are trivial. For example, in an axiomatic proof, proof of soundness amounts to verifying the validity of the axioms and that the rules of inference preserve validity. On the contrary, completeness property is much harder in general. Soundness and completeness theorems link the syntax and semantics of modal logics, by providing a correspondence between provability (\vdash) and validity (\models).

In order to prove soundness and completeness of some well-known normal modal logics we shall need the following fundamental concept.

Definition 1.5.1 (Consistency) Let L be a normal modal logic. A set S of formulas of propositional modal logic is called L -consistent if and only if there are no formulas $\varphi_1, \dots, \varphi_n \in S$ with $(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n) \rightarrow \perp \in L$

,where \perp means 'falsity'. Otherwise, S is called L -inconsistent. A consistent set S of propositional modal formulas is called '*maximally consistent*' if and only if for every formula ϕ either $\phi \in S$ or $\neg \phi \in S$.

We assume normal modal logics L to be consistent.

Lemma 1.5.2

Let L be a normal modal logic and S maximally consistent. Then

- (i) For every formula ϕ exactly one of the following cases holds : either $\phi \in S$ or $\neg \phi \in S$.
- (ii) $\phi \in S$, $\phi \rightarrow \psi \in S$ implies that $\psi \in S$ (closed under modus ponens).
- (iii) $\phi \wedge \psi \in S$ if and only if $\phi \in S$ and $\psi \in S$.
- (iv) $\phi \vee \psi \in S$ if and only if $\phi \in S$ or $\psi \in S$.
- (v) $L \subseteq S$.

Proof

(i) If both were in S , then S would be inconsistent, because the propositional tautology $\phi \wedge \neg \phi \rightarrow \perp \in L$.

(ii) Assume that $\phi \in S$, $\phi \rightarrow \psi \in S$, but $\psi \notin S$. By maximal consistency, $\neg \psi \in S$. Now consider the propositional tautology $(\phi \wedge (\phi \rightarrow \psi) \wedge \neg \psi) \rightarrow \perp \in L$. This contradicts the consistency of S .

(iii) similar to (iv).

(iv) **CN** (\Rightarrow) : Let $\phi \vee \psi \in S$ and assume that $\phi \notin S$ and $\psi \notin S$. Hence by maximal consistency of S , $\neg \phi \in S$ and $\neg \psi \in S$. Also the tautology $((\neg \phi \wedge \neg \psi) \wedge (\phi \vee \psi)) \rightarrow \perp \in L$. That contradicts the consistency of S .

CS (\Leftarrow) : Let $\phi \in S$ and $\phi \vee \psi \notin S$. Then maximal consistency shows that $\neg(\phi \vee \psi) \in S$. But the tautology $(\phi \wedge (\phi \vee \psi)) \rightarrow \perp \in L$ contradicts the consistency of S .

(v) Let $\phi \in L$. Then $\{\neg \phi\}$ is L -inconsistent. Thus $\neg \phi \notin S$. By maximal consistency, $\phi \in S$; hence $L \subseteq S$. \otimes

Lemma 1.5.3 (Lindenbaum's Lemma)

For every consistent set S there is a maximally consistent superset M , or equivalently every consistent set can be enlarged to a maximally consistent set.

Proof (see Chellas, p.55, for example)

Another important concept in modal logic is the concept of canonicity.

Definition 1.5.4 Let L be a normal modal logic. The 'canonical model' M^L is the triple $\langle W^L, R^L, V^L \rangle$ where

- (i) W^L is the set of all maximal consistent sets of formulas;
- (ii) R^L is the binary relation defined on W^L by $w R^L u$ if for all formulas $\varphi, \psi \in u$ implies $\Box \varphi \in w$. R^L is called the 'canonical relation';
- (iii) V^L is the valuation defined by $V^L(p) = \{ w \in W^L \mid p \in w \}$. V^L is called the 'canonical (or natural) valuation'.

The pair $F^L = \langle W^L, R^L \rangle$ is called the 'canonical frame' for L .

Lemma 1.5.5 (Truth Lemma)

For a normal propositional modal logic L , let $M^L = \langle W^L, R^L, V^L \rangle$ be the canonical model of L . Then for any world $S \in W^L$ and any formula φ , $M^L, S \models \varphi$ if and only if $\varphi \in S$.

Proof (see Blackburn, Rijke and Venema, p.199)

Corollary 1.5.6

Let M^L be the canonical model of L . Then $\varphi \in L$ if and only if $M^L \models \varphi$.

Proof

CN (\Rightarrow): By Lemma 1.5.2 (v), we have that L is a subset of every world $S \in W^L$. Thus necessary condition follows from Truth Lemma.

CS (\Leftarrow) : Let $M^L \models \varphi$, i.e. $M^L, S \models \varphi$, for all $S \in W^L$. Suppose that $\varphi \notin L$. But then $L \cup \{\neg\varphi\}$ would be consistent; otherwise there were $\varphi_1 \dots \varphi_n \in L$ with $(\varphi_1 \wedge \dots \wedge \varphi_n \wedge \neg\varphi) \rightarrow \perp \in L$ which would imply $\varphi \in L$. Since $L \cup \{\neg\varphi\}$ is consistent, there, thus, is a maximally consistent extension $T \in W^L$ with $T \supseteq L \cup \{\neg\varphi\}$, by Lindenbaum's Lemma. In particular, $\neg\varphi \in T$, such that Lemma 1.5.5 implies $M^L, T \models \neg\varphi$, which would contradict $M^L \models \varphi$. \otimes

This implies a kind of completeness, but is surprising in that it connects provability in a system with validity, not in all, but in one model.

Corollary 1.5.7 Let \vdash_s be a provability relation for a normal modal logic proof system and M^L the canonical model for the logic $L := \{\varphi : \vdash_s \varphi\}$. Then $\vdash_s \varphi$ if and only if $M^L \models \varphi$.

Proof Consider $L := \{\varphi : \vdash_s \varphi\}$ in corollary 1.5.6. \otimes

This corollary is a starting point for proving full completeness.

Theorem 1.5.8 (Completeness for **K**)

For every modal logic formula φ

$$\vdash_{\mathbf{K}} \varphi \text{ if and only if } \models_{\mathbf{K}} \varphi \text{ if and only if } M \models \varphi \text{ for every model } M.$$

Proof If $M \models \varphi$ for every model M , then also for the canonical model, thus corollary 1.5.7 implies $\vdash_{\mathbf{K}} \varphi$. The converse direction is soundness that every axiom of **K** holds in all models and every proof rule of **K** preserves validity. \otimes

Theorem 1.5.9 (Completeness for **T**)

For every modal logic formula φ

$$\vdash_{\mathbf{T}} \varphi \text{ if and only if } \models_{\mathbf{T}} \varphi \text{ for every modal } M.$$

Proof The only new part is the need to show that the T-axiom is true in all reflexive models, and that the canonical model for **T** is reflexive. Consider a maximal **T**-consistent set S . We have to show that $\{\varphi : \Box\varphi \in S\} \subseteq S$. Consider any $\Box\varphi \in S$. Now by Lemma 1.5.2, the T-instance $\Box\varphi \rightarrow \varphi$ is an element of S , thus $\varphi \in S$. \otimes

In a similar way, completeness of many normal modal logics, such as **S4** and **S5**, can be shown [3].

Chapter 2

TOPOLOGICAL SEMANTICS

In order to understand the content of the present chapter, we need to know basic concepts of general topology.

2.1 Basic notions of general topology

Definition 2.1.1 Let X be a set and τ a subset of the powerset of X . τ is a topology on X if

(i) $\emptyset, X \in \tau$,

(ii) If $U_i \in \tau$ with $1 \leq i \leq n$, then $\bigcap_{i=1}^n U_i \in \tau$

(iii) Any arbitrary union of elements of τ is an element of τ .

The pair $\langle X, \tau \rangle$ is called a topological space.

Definition 2.1.2 Let $\langle X, \tau \rangle$ be a topological space, and let Y be a subset of X . Y is said to be an open set of the space, or an open subset of X if Y is an element of τ .

From now on, we can write $\langle X, \tau \rangle$ simply by T .

Definition 2.1.3 Let $T = \langle X, \tau \rangle$ be a space and let Y be a subset of X . Y is a closed set of T , or Y is closed in T if $Y^c = X - Y$ is open in T .

Clearly a space T is finite if X is finite.

Definition 2.1.4 Let $T_1 = \langle X_1, \tau_1 \rangle$ be a space and let X_2 be a nonempty subset of X_1 . A subset τ_2 of $P(X_2)$ is the relative topology of T_1 to X_2 if $\tau_2 = \{Y_2 \in P(X_2) : \exists O_1 \in \tau_1 (Y_2 = O_1 \cap X_2)\}$

In fact, the relative topology of T_1 to X_2 is a topology on X_2 .

Definition 2.1.5 Let $T_1 = \langle X_1, \tau_1 \rangle$ be a space, and let X_2 be a nonempty subset of X_1 . A space $T_2 = \langle X_2, \tau_2 \rangle$ is a subspace of T_1 if τ_2 is the relative topology of T_1 to X_2 . A subspace $T_2 = \langle X_2, \tau_2 \rangle$ of T_1 is an open subspace of T_1 if X_2 is open in T_1 .

Definition 2.1.6 Let $T = \langle X, \tau \rangle$ be a space, and let Y be a subset of X .

. The interior $\mathbf{I}(Y)$, or $\text{Int}(Y)$ of Y is a subset of X such that

$$\mathbf{I}(Y) = \{x \in X : \exists O \in \mathcal{T}(x \in O \text{ and } O \subseteq Y)\}.$$

The closure $\mathbf{C}(Y)$, or $\mathbf{Cl}(Y)$, of Y is a subset of X such that

$$\mathbf{C}(Y) = \{x \in X : \forall O \in \mathcal{T}(x \in O \Rightarrow O \cap Y \neq \emptyset)\}$$

It is known that $\mathbf{I}(Y)$ is the largest open set contained in Y , while $\mathbf{C}(Y)$ is the smallest closed set containing Y .

. The frontier $\mathbf{Fr}(Y)$, or the boundary of Y is a subset of X such that

$$\mathbf{Fr}(Y) = \mathbf{C}(Y) \cap \mathbf{C}(X-Y).$$

The following results can be easily verified.

Proposition 2.1.7 Let $\langle X, \mathcal{T} \rangle$ be a space, and let Y and Z be subset of X . Then the following hold.

- (1) $\mathbf{I}(Y) \in \mathcal{T}$
- (2) $\mathbf{I}(Y) \subseteq Y$.
- (3) $(Z \in \mathcal{T} \text{ and } Z \subseteq Y) \Rightarrow Z \subseteq \mathbf{I}(Y)$.
- (4) $\mathbf{I}(Y) = Y \Leftrightarrow Y \in \mathcal{T}$
- (5) $\mathbf{I}(\mathbf{I}(Y)) = \mathbf{I}(Y)$.
- (6) $\mathbf{I}(Y \cap Z) = \mathbf{I}(Y) \cap \mathbf{I}(Z)$.
- (7) $Y \subseteq Z \Rightarrow \mathbf{I}(Y) \subseteq \mathbf{I}(Z)$
- (8) $\mathbf{I}(Y) = X - \mathbf{C}(X - Y)$. \otimes

It follows from (5), (6), and (7) that the operator \mathbf{I} is idempotent, preserves the operation intersection and is monotone, respectively.

Proposition 2.1.8 Let $\langle X, \mathcal{T} \rangle$ be a space, and let Y and Z be subsets of X . Then the following hold.

- (1) $X - \mathbf{C}(Y) \in \mathcal{T}$
- (2) $Y \subseteq \mathbf{C}(Y)$
- (3) $(X - Z \in \mathcal{T} \text{ and } Y \subseteq Z) \Rightarrow \mathbf{C}(Y) \subseteq Z$.
- (4) $\mathbf{C}(Y) = Y \Leftrightarrow X - Y \in \mathcal{T}$
- (5) $\mathbf{C}(\mathbf{C}(Y)) = \mathbf{C}(Y)$.
- (6) $\mathbf{C}(Y \cup Z) = \mathbf{C}(Y) \cup \mathbf{C}(Z)$
- (7) $Y \subseteq Z \Rightarrow \mathbf{C}(Y) \subseteq \mathbf{C}(Z)$

$$(8) C(Y) = X - I(X - Y).$$

Note that the operator \mathbf{C} is idempotent, preserves the operation union and is montone by (5),(6), and (7), respectively. Also \mathbf{I} and \mathbf{C} are interdefinable by (8).

On the other hand the notations $\overset{0}{A}$ and $\overset{-}{A}$ are used for $\text{int}(A)$ and $\text{Cl}(A)$.

Definition 2.1.9 A space $\langle X, \tau \rangle$ is connected if τ is connected, that is,

$$\forall Y \subseteq X ((Y \in \tau \text{ and } X - Y \in \tau) \Rightarrow (Y = \emptyset \text{ or } Y = X))$$

A space $\langle X, \tau \rangle$ is disconnected if it is not connected, that is,

$$\exists Y \subseteq X ((Y \in \tau \text{ and } X - Y \in \tau) \text{ and } (Y \neq \emptyset \text{ and } Y \neq X))$$

Definition 2.1.10 A space $\langle X, \tau \rangle$ is extrenally disconnected if τ is extrenally disconnected, that is,

$$\forall O \in \tau (C(O) \in \tau)$$

Definition 2.1.11 Let $T = \langle X, \tau \rangle$ be a space.

(i) T is a T_0 - space if $\forall x, \forall y \in X$

$$x \neq y \Rightarrow \exists O \in \tau (x \in O \text{ and } y \notin O) \text{ or } \exists O \in \tau (y \in O \text{ and } x \notin O)$$

(ii) T is a T_1 - space if $\forall x, \forall y \in X$

$$x \neq y \Rightarrow \exists O \in \tau (x \in O \text{ and } y \notin O) \text{ and } \exists O \in \tau (y \in O \text{ and } x \notin O)$$

(iii) T is a T_2 - space, or a Hausdorff space if

$$\forall x, \forall y \in X (x \neq y \Rightarrow \exists u \in \tau, \exists O \in \tau (x \in U \text{ and } y \in O \text{ and } U \cap O = \emptyset))$$

Note the implications : $T_2 \Rightarrow T_1 \Rightarrow T_0$

Proposition 2.1.12 A space $\langle X, \tau \rangle$ is a T_1 - space if and only if

$$\forall x \in X (X - \{x\} \in \tau). \otimes$$

Let us recall some other topologies.

. An Alexandroff space is a topological space for which arbitrary intersections of open sets are also open sets.

. A discrete space is a space for which $\tau = P(X)$.

. An indiscrete space is a space for which τ consists of only \emptyset and X .

. A space $\langle X, \tau \rangle$ is atomic if

$$\forall Y \subseteq X (I(\text{Fr}(Y)) = \emptyset).$$

These well-known concepts can be found in any general textbooks on point set topology. See for example ‘Engelking, R., General Topology, Heldermann Verlag, 1989’.

2.2 The Basic Modal Language and Topological Semantics

We have presented an overview of Kripke semantics, or relational semantics, in chapter 1, based on the propositional modal language. This chapter provides a different semantics of this language, which is called topological semantics based on frameworks by topological spaces.

Topological Models

In topological semantics, topological spaces play roles similar to Kripke frames in Kripke semantics. Also, topological models correspond to Kripke models.

Definition 2.2.1 A topological model on a space $T = \langle X, \tau \rangle$ is a structure

$\langle X, \tau, V \rangle$ with a map V from At to $P(X)$. The map V is called a valuation on T . As in the case of Kripke semantics, a relation ‘a formula ϕ is true at a point x in a topological model $M = \langle X, \tau, V \rangle$ ’, denoted

$$M, x \models \phi$$

is inductively defined as follows :

- (i) $\forall p \in At \quad \forall x \in X (M, x \models p \Leftrightarrow x \in V(p))$
- (ii) $\forall x \in X (M, x \not\models \perp)$
- (iii) $\forall \phi, \forall \Psi \in For \quad \forall x \in X (M, x \models \phi \rightarrow \Psi \Leftrightarrow (M, x \models \phi \Rightarrow M, x \models \Psi))$, where F for is the set of formulas of modal language,
- (iv) $\forall \phi \in For \quad \forall x \in X (M, x \models \Box \phi \Leftrightarrow \exists O \in \tau (x \in O \text{ and } \forall y \in O (M, y \models \phi)))$

When a context makes it clear, we often abbreviate topological models to models.

Proposition 2.2.2 Let $M = \langle X, \tau, V \rangle$ be a model. Then the following hold.

- (1) $\forall \phi, \forall \Psi \in For \quad \forall x \in X (M, x \models \phi \wedge \Psi \Leftrightarrow (M, x \models \phi \text{ and } M, x \models \Psi))$
- (2) $\forall \phi, \forall \Psi \in For \quad \forall x \in X (M, x \models \phi \vee \Psi \Leftrightarrow (M, x \models \phi \text{ or } M, x \models \Psi))$
- (3) $\forall \phi \in For \quad \forall x \in X (M, x \models \neg \phi \Leftrightarrow M, x \not\models \phi)$
- (4) $\forall x \in X (M, x \models T)$
- (5) $\forall \phi \in For \quad \forall x \in X (M, x \models \Diamond \phi \Leftrightarrow \forall O \in \tau (x \in O \Rightarrow \exists y \in O (M, y \models \phi)))$

Proof The only interesting part is (5) ; the remaining ones are obvious. So let φ be a formula and let x be an element of X . Then

$$\begin{aligned}
& M, x \models \diamond\varphi \\
& \Leftrightarrow M, x \models \neg\Box\neg\varphi && \text{(by } \diamond \equiv \neg\Box\neg) \\
& \Leftrightarrow M, x \not\models \Box\neg\varphi && \text{(by (3))} \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow \exists y \in O (M, y \not\models \neg\varphi)) && \text{(by negation of } \models\Box\varphi) \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow \exists y \in O (M, y \models \varphi)) && \text{(by (3)). } \otimes
\end{aligned}$$

2.3 Truth and Validity

Truth and validity of formulas in models are ‘almost’ the same as those in Kripke semantics.

Definition 2.3.1 (Truth in models)

Let $M = \langle X, \mathcal{T}, V \rangle$ be a model and let φ be a formula φ is true in M , denoted $M \models \varphi$, if $\forall x \in X (M, x \models \varphi)$, φ is false in M if it is not true in M .

Definition 2.3.2 (Validity in Spaces)

Let φ be a formula, let $T = \langle X, \mathcal{T} \rangle$ be a space, and let \mathcal{T} be a class of frames. φ is valid in T , or T validates φ , denoted $T \models \varphi$, if

$$\forall V \in \text{Map}(At, P(X)) ((X, \mathcal{T}, V) \models \varphi)$$

φ is false in T , or T falsities φ if φ is not valid in T . φ is valid in \mathcal{T} , denoted $\mathcal{T} \models \varphi$, if $\forall T \in \mathcal{T} (T \models \varphi)$

In the following proposition formulas are interpreted in topological spaces. All parts of this proposition, except for the last two, are the same as those in Kripke models. The modal operators \Box and \diamond get interpreted as the interior operator **I** and the closure operator **C**, respectively.

Proposition 2.3.3 Let $M = \langle X, \mathcal{T}, V \rangle$ be a model. By definition of V ,

$$\forall p \in At (V(p) = \{x \in X : M, x \models p\}),$$

and extended to formulas,

$\forall \varphi \in For (V(\varphi) = \{x \in X : M, x \models \varphi\})$ Then the following hold, for all $\varphi, \psi \in For$:

- (1) $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$
- (2) $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$

- (3) $V(\neg\varphi) = X - V(\varphi)$
- (4) $V(\varphi \rightarrow \psi) = (X - V(\varphi)) \cup V(\psi)$
- (5) $M \models \varphi \rightarrow \beta \Leftrightarrow V(\varphi) \subseteq V(\beta)$
- (6) $V(\top) = X$
- (7) $V(\perp) = \emptyset$
- (8) $V(\Box\varphi) = \mathbf{I}(V(\varphi))$
- (9) $V(\Diamond\varphi) = \mathbf{C}(V(\varphi))$

Proof We only prove parts (8) and (9) since the others have the same arguments for Kripke semantics.

(8) : As sets, we show that $V(\Box\varphi)$ and $\mathbf{I}(V(\varphi))$ are equal. So

$$\begin{aligned}
& x \in V(\Box\varphi) \\
& \Leftrightarrow M, x \models \Box\varphi && \text{(by definition of } V(u)) \\
& \Leftrightarrow \exists O \in \mathcal{T}(x \in O \text{ and } \forall y \in O(M, y \models \varphi)) && \text{(Definition 2.2.1(iv))} \\
& \Leftrightarrow \exists O \in \mathcal{T}(x \in O \text{ and } \forall y \in O(y \in V(\varphi))) && \text{(Definition of } V(u)) \\
& \Leftrightarrow \exists O \in \mathcal{T}(x \in O \text{ and } \forall y \in X(y \in O \Rightarrow y \in V(\varphi))) \\
& \Leftrightarrow \exists O \in \mathcal{T}(x \in O \text{ and } O \subseteq V(\varphi)) \\
& \Leftrightarrow x \in \mathbf{I}(V(\varphi)) && \text{(Definition of } \mathbf{I})
\end{aligned}$$

(Thus, $x \in V(\Box\varphi) \Leftrightarrow x \in \mathbf{I}(V(\varphi))$ implies that $V(\Box\varphi) = \mathbf{I}(V(\varphi))$)

(9) By similar argument as in (8), we have

$$\begin{aligned}
& x \in V(\Diamond\varphi) \\
& \Leftrightarrow M, x \models \Diamond\varphi && \text{(Definition of } V(U)) \\
& \Leftrightarrow M, x \models \neg\Box\neg\varphi && \text{(By } \Diamond \equiv \neg\Box\neg) \\
& \Leftrightarrow M, x \not\models \Box\neg\varphi && \text{(Prop. 2.2.2 (3))} \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow \exists y \in O(M, y \not\models \neg\varphi)) && \text{(Negation of Def.2.2.1(iv))} \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow \exists y \in O(M, y \models \varphi)) && \text{(Prop.2.2.2(3))} \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow \exists y \in O(y \in V(\varphi))) && \text{(Def. of } V(U)) \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow \exists y \in X(y \in O \text{ and } y \in V(\varphi))) \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow \exists y \in X(y \in O \cap V(\varphi))) \\
& \Leftrightarrow \forall O \in \mathcal{T}(x \in O \Rightarrow O \cap V(\varphi) \neq \emptyset) \\
& \Leftrightarrow x \in \mathbf{C}(V(\varphi)).
\end{aligned}$$

Thus $x \in V(\diamond\varphi) \Leftrightarrow x \in \mathbf{C}(V(\varphi))$, hence we have the required equality. \otimes

Soundness and completeness are defined exactly the same way as in these in Kripke semantics. We will also need ‘definability’ concept, which is of great importance for the topological semantics.

Definition 2.3.4 Let \mathcal{T} be a class of spaces. A set of formulas Σ defines \mathcal{T} , if for any space T , $T \in \mathcal{T} \Leftrightarrow \forall \varphi \in \Sigma (T \models \varphi)$

Definition 2.3.5 Let L be a logic, and let \mathcal{T} be a class of spaces. L is sound with respect to \mathcal{T} if

$$\forall \varphi \in \text{For} (\vdash_L \varphi \Rightarrow \mathcal{T} \models \varphi)$$

L is complete with respect to \mathcal{T} if

$$\forall \varphi \in \text{For} (\mathcal{T} \models \varphi \Rightarrow \vdash_L \varphi)$$

2.4 Relationships between Kripke Frames and Topological Spaces

In this section, we deal with issues between Kripke semantics and topological semantics. To start, we give

Definition 2.4.1 Let $F = \langle X, R \rangle$ be a reflexive and transitive frame. A subset Y of X is said to be upward closed if

$$\forall x, \forall y \in X ((x \in Y \text{ and } xRy) \Rightarrow y \in Y)$$

Proposition 2.4.2 Let U be the set of all upward closed sets of a reflexive and transitive frame $F = \langle X, R \rangle$. Then U is a topology on X , i.e., $T = \langle X, U \rangle$ is a topological space. Moreover, T is an Alexandroff space.

Proof That X and \emptyset are upward closed sets is obvious. Now let $\{Y_i\}_{i \in I}$ be a family of upward closed sets. Then for any elements $x, y \in X$, we have

$$x \in \bigcup_{i \in I} Y_i \text{ and } xRy$$

$$\Leftrightarrow \exists j \in I (x \in Y_j) \text{ and } xRy \quad (\text{by def. of } U)$$

$$\Leftrightarrow \exists j \in I (y \in Y_j) \quad (Y_j \text{ is upward closed})$$

$$\Leftrightarrow y \in \bigcup_{i \in I} Y_i \quad (\text{def. of } U)$$

,so $\bigcup_{i \in I} Y_i$ is also upward closed.

Now $x \in \bigcap_{i \in I} Y_i$ and xRy

$\Leftrightarrow \forall j \in I (x \in Y_j)$ and xRy (def. of \bigcap)

$\Leftrightarrow \forall j \in I (y \in Y_j)$ (Y_j is upward closed)

$\Leftrightarrow y \in \bigcap_{i \in I} Y_i,$

thus $\bigcap_{i \in I} Y_i$ is also an upward closed set. Hence, as the intersection is arbitrary, $T = \langle X, U \rangle$ is an Alexandroff space. \otimes

Clearly, for each reflexive and transitive frame F , the set U uniquely determines an Alexandroff space. We call such a space ‘the corresponding space’ to F . The next result is an important fact.

Proposition 2.4.3 Let $T = \langle X, U \rangle$ be the corresponding space to the reflexive and transitive frame $F = \langle X, R \rangle$. Then for every valuation $V, x \in X$ and $\varphi \in For$,

$\langle X, R, V \rangle, x \models \varphi \Leftrightarrow \langle X, U, V \rangle, x \models \varphi$.

Proof We prove by induction on the complexity of formula φ . The Boolean cases are easy to verify. We only prove for the modal connective \Box . Let

$M_1 = \langle X, R, V \rangle$ be a model on F , and let $M_2 = \langle X, U, V \rangle$ be a model on T . For a formula Ψ , assume that

$$\forall x \in X (M_1, x \models \Psi \Leftrightarrow M_2, x \models \Psi)$$

We have to establish that

$$\forall x \in X (M_1, x \models \Box \Psi \Leftrightarrow M_2, x \models \Box \Psi)$$

Now $M_1, x \models \Box \Psi$

$\Leftrightarrow \forall y \in X (xRy \Rightarrow M_1, y \models \Psi)$ (By def. of \Box)

$\Leftrightarrow \forall y \in X (xRy \Rightarrow M_2, y \models \Psi)$ (By hypothesis)

$\Leftrightarrow \exists O \in U (x \in O \text{ and } \forall y \in O (M_2, y \models \Psi))$

$\Leftrightarrow M_2, x \models \Box \Psi . \otimes$

This proposition shows that the corresponding space to any reflexive and transitive frame preserves the validity of any formula. This result will be used to prove completeness of some logics. Another fact used to establish completeness is the following proposition which gives an explanation how the interior operator **I** and the closure operator **C** in the corresponding space to a given reflexive and transitive frame are represented in the frame.

Proposition 2.4.4 Let $T = \langle X, U \rangle$ be the corresponding space to a reflexive and transitive frame $F = \langle X, R \rangle$. Then for any subset Y of X ,

$$\mathbf{I}(Y) = \left\{ x \in X \mid \forall y \in X (xRy \Rightarrow y \in Y) \right\},$$

$$\mathbf{C}(Y) = \left\{ x \in X \mid \exists y \in X (xRy \text{ and } y \in Y) \right\}.$$

Proof Let x be an element of X . Then

$$\begin{aligned} x \in \mathbf{I}(Y) \\ \Leftrightarrow \exists O \in U (x \in O \text{ and } O \subseteq Y) \\ \Leftrightarrow \exists O \in U (\forall y \in X (xRy \Rightarrow y \in O) \text{ and } O \subseteq Y) \\ \Leftrightarrow \forall y \in X (xRy \Rightarrow y \in Y) \end{aligned}$$

and

$$\begin{aligned} x \in \mathbf{C}(Y) \\ \Leftrightarrow x \in X - \mathbf{I}(X - Y) \\ \Leftrightarrow x \notin \mathbf{I}(X - Y) \\ \Leftrightarrow \exists x \in X (xRy \text{ and } y \notin X - Y) \\ \Leftrightarrow \exists x \in X (xRy \text{ and } y \in Y) . \otimes \end{aligned}$$

2.5 Definability of Topological Spaces

In this section, we will prove that some topological spaces are definable. For those not definable in the basic modal language, see [4].

Lemma 2.5.1 A topological space $T = \langle X, \tau \rangle$ is discrete if and only if $\varphi \rightarrow \Box\varphi$ is valid in T .

Proof Condition is necessary : Assume that T is discrete and let $M = \langle X, \tau, V \rangle$ be a model on T . We have to prove that

$$M \models \varphi \rightarrow \Box\varphi$$

Since T is discrete, $V(\varphi)$ is open, i.e.,

$$V(\varphi) = \mathbf{I}(V(\varphi)).$$

Now $V(\varphi) = \mathbf{I}(V(\varphi))$

$$\Leftrightarrow V(\varphi) \subseteq \mathbf{I}(V(\varphi))$$

$$\Leftrightarrow V(\varphi) \subseteq V(\Box\varphi)$$

$$\Leftrightarrow M \models \varphi \rightarrow \Box \varphi.$$

Condition is sufficient : Assume that $\varphi \rightarrow \Box \varphi$ is valid in T . Prove that T is discrete. To see this, let Y be any subset of X . Then it suffices to show that Y is open, i.e.

$$Y = \mathbf{I}(Y)$$

Now, since $\varphi \rightarrow \Box \varphi$ is valid in T , we have in particular $M \models p \rightarrow \Box p$ for $p \in At$ and such that $V(p)=Y$, with our model $M = \langle X, \tau, V \rangle$. Then

$$\begin{aligned} M \models p \rightarrow \Box p \\ \Leftrightarrow V(p) \subseteq V(\Box p) \\ \Leftrightarrow V(p) \subseteq \mathbf{I}(V(p)) \\ \Leftrightarrow Y \subseteq \mathbf{I}(Y) \\ \Leftrightarrow Y = \mathbf{I}(Y). \quad \otimes \end{aligned}$$

Theorem 2.5.2 The schema $\varphi \rightarrow \Box \varphi$ defines the class of all discrete spaces. \otimes

Atomic topology is not so popular in general topology, however the schema that defines the class of atomic topologies is well known. The formula (or schema) $\Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$ is valid in atomic spaces and atomic spaces validate the formula. More precisely, we have

Lemma 2.5.3 A topological space is atomic if and only if it validates the schema $\Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$.

Proof Condition is sufficient. Let $T = \langle X, \tau \rangle$ be a topological space and assume that $\Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$ is valid in T . We prove that T is atomic. To see this, let Y be a subset of X . Since the formula is valid, then there exist $p \in At$ and a valuation V such that

$$V(p)=Y, \quad \langle X, \tau, V \rangle \models \Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$$

Put $M = \langle X, \tau, V \rangle$. Now

$$\begin{aligned} M \models \Box \Diamond \varphi \rightarrow \Diamond \Box \varphi \\ \Leftrightarrow V(\Box \Diamond p) \subseteq V(\Diamond \Box p) \\ \Leftrightarrow \mathbf{I}(V(\Diamond p)) \subseteq \mathbf{C}(V(\Box p)) \\ \Leftrightarrow \mathbf{I}(\mathbf{C}(V(p))) \subseteq \mathbf{C}(\mathbf{I}(V(p))) \\ \Leftrightarrow \mathbf{I}(\mathbf{C}(Y)) \subseteq \mathbf{C}(\mathbf{I}(Y)) \quad (\text{since } V(p)=Y) \\ \Leftrightarrow \mathbf{I}(\mathbf{C}(Y)) \cap (X - \mathbf{C}(\mathbf{I}(Y))) = \emptyset \\ \Leftrightarrow \mathbf{I}(\mathbf{C}(Y)) \cap (X - (X - \mathbf{I}(X - \mathbf{I}(Y)))) = \emptyset \\ \Leftrightarrow \mathbf{I}(\mathbf{C}(Y)) \cap \mathbf{I}(X - \mathbf{I}(Y)) = \emptyset \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \mathbf{I}(\mathbf{C}(Y)) \cap \mathbf{I}(X - (X - \mathbf{C}(X - Y))) = \emptyset \\
&\Leftrightarrow \mathbf{I}(\mathbf{C}(Y)) \cap \mathbf{I}(\mathbf{C}(X - Y)) = \emptyset \\
&\Leftrightarrow \mathbf{I}(\mathbf{C}(Y)) \cap (\mathbf{C}(X - Y)) = \emptyset \\
&\Leftrightarrow \mathbf{I}(\text{Fr}(Y)) = \emptyset,
\end{aligned}$$

proving that T is atomic.

Condition is necessary : Assume that $T = \langle X, \mathcal{T} \rangle$ is atomic and show that $\Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$ is valid on a model $M = \langle X, \mathcal{T}, V \rangle$. Since T is atomic, we have $\mathbf{I}(\text{Fr}(V(\varphi))) = \emptyset$. Now by the latter part of the proof of sufficiency, we obtain the following equivalences :

$$\begin{aligned}
&\mathbf{I}(\text{Fr}(V(\varphi))) = \emptyset \\
&\Leftrightarrow \mathbf{I}(\mathbf{C}(V(\varphi))) \subseteq \mathbf{C}(\mathbf{I}(V(\varphi))) \\
&\Leftrightarrow (\Box \Diamond \varphi) \subseteq V(\Diamond \Box \varphi) \\
&\Leftrightarrow M \models \Box \Diamond \varphi \rightarrow \Diamond \Box \varphi. \otimes
\end{aligned}$$

Theorem 2.5.4 The schema $\Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$ defines the class of all atomic spaces.

\otimes

For more definable spaces, see [14].

2.6 Soundness and Completeness

In this last section, we will illustrate soundness and completeness of only S4 and S4.1, recalling that

$S4 = T \oplus 4$, where $T: \Box \varphi \rightarrow \varphi$ and $4: \Box \varphi \rightarrow \Box \Box \varphi$ and $S4.1 = S4 \oplus M$, where $M = \Box \Diamond \varphi \rightarrow \Diamond \Box \varphi$.

2.6.1 Soundness and Completeness of S4

This subsection provides one of the most important results in topological semantics. McKinsey and Tarski's work [8] implicitly mention it.

Theorem 2.6.1.1 S4 is sound with respect to the class of all topological spaces.

Proof It is easy to verify that all classical tautologies and the schemata $K, T, 4$ are valid in any topological space. On the other hand, it is well-known that the rules modus ponens and Necessitation preserve the validity of any formula. \otimes

Theorem 2.6.1.2 S4 is complete with respect to the class of all topological spaces.

Proof Let φ be a formula which is a nontheorem of S4. We prove that there exists a space which falsifies φ . By the completeness of S4 in Kripke semantics, there exists a reflexive and transitive frame $F = \langle X, R \rangle$ such that

$$\exists V : For \rightarrow P(X), \exists x \in X, (\langle X, R, V \rangle, x \not\models \varphi)$$

Take the corresponding space $T = \langle X, U \rangle$. Then

$$\exists V : For \rightarrow P(X), \exists x \in X, (\langle X, U, V \rangle, x \not\models \varphi)$$

Thus T falsifies φ . \otimes

S4 has the finite frame property in Kripke semantics. Therefore, it has the finite space property in topological semantics.

Theorem 2.6.1.3 S4 is sound and complete with respect to the class of all finite topological spaces. \otimes

2.6.2 Soundness and Completeness of S4.1

S4.1 is an extension of S4, and it is, as we prove below, sound and complete with respect to the class of all atomic spaces. Its soundness is obvious.

Theorem 2.6.2.1 S4.1 is sound with respect to the class of all atomic spaces. \otimes

As usual, completeness is not easy to prove. To show the completeness of S4.1, we need a lemma.

Lemma 2.6.2.2 The corresponding space to a reflexive, transitive and atomic frame is atomic.

Proof Let $T = \langle X, U \rangle$ be the corresponding space to a reflexive, transitive and atomic frame $F = \langle X, R \rangle$, and let Y be a subset of X . We shall prove that

$$\mathbf{I}(\text{Fr}(Y)) = \emptyset$$

To see it, for $x \in X$, it suffices to show that

$$x \notin \mathbf{I}(\text{Fr}(Y))$$

In the content of frames, R is said to be atomic if

$$\exists y \in X (xRy \text{ and } \forall z \in X (yRz \Rightarrow y = z))$$

is satisfied. Using this definition and the relations between the operators \mathbf{I} and \mathbf{C} , easy computations lead to the desired conclusion $x \notin \mathbf{I}(\text{Fr}(Y))$ or any $x \in X$, hence F is atomic. \otimes

Theorem 2.6.2.3 S4 is complete with respect to the class of all atomic spaces.

Proof Let a formula φ be a nontheorem of S4.1. We shall prove that there exists an atomic space falsifying φ . Again, by the completeness of S4.1 in Kripke semantics, there exists a reflexive, transitive and atomic frame $F = \langle X, R \rangle$ such that

$$\exists V : For \rightarrow P(X), \exists x \in X (\langle X, R, V \rangle, x \not\models \varphi).$$

Take the corresponding space $T = \langle X, U \rangle$. Then T is atomic and

$$\exists V : For \rightarrow P(X), \exists x \in X (\langle X, U, V \rangle, x \not\models \varphi).$$

Thus, T falsifies φ . \otimes

For more completeness, see [14].

Chapter 3

NEIGHBORHOOD SEMANTICS FOR MODAL LOGIC

Neighborhood semantics is a generalization of the Kripke or relational semantics. K. Segerberg published an essay [13] presenting some basic results about neighborhood models and the classical systems that correspond to them. Later on B. Chellas incorporated these and other outstanding results in his textbook [2]. In fact, the idea for neighborhood semantics for modal logic is already implicit in the seminal work of McKinsey and Tarski [8].

The goal in this chapter is to understand the techniques, results and applications of neighborhood semantics for modal logic and to understand the exact relationship with the Kripke semantics.

3.1 Basic Concepts

3.1.1 Preliminaries

Let W be a nonempty set and $P(W)$ the collection of all subsets of W . We are interested in pairs $\langle W, F \rangle$ where $F \in P(P(W))$ or $F \subseteq P(W)$. The sets F satisfy certain algebraic properties that we list below.

1. F is closed under intersections if for any collection of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in F$, then $\bigcap_{i \in I} X_i \in F$. If $|I| = 2$, then we say F is closed under binary intersections. If I is finite, then we say F is closed under finite intersections. Similarly, the **closure under unions** can be defined.
2. F is **closed under complements** if for each $X \subseteq W$, if $X \in F$, then $X^c \in F$.
3. F is **supplemented**, or **closed under supersets** or an **up-set** provided for each $X \subseteq W$, if $X \in F$ and $X \subseteq Y \subseteq W$, then $Y \in F$.
4. F **contains the unit** provided $W \in F$; and F **contains the empty set** if $\emptyset \in F$.
5. Call the set $\bigcap_{X \in F} X$ the core of F . F **contains its core** if $\bigcap_{X \in F} X \in F$.
6. F is **proper** if $X \in F$ implies $X^c \notin F$.

7. F is **consistent** if $\emptyset \notin F$ and $F \neq \emptyset$.

The following result provides an alternative characterization of up-set collections, whose proof is tricial.

Lemma 3.1.2 F is supplemented if and only if, if $X \cap Y \in F$, then $X \in F$ and $Y \in F$. \otimes

Remark If F contains the unit, then F contains a maximal element (under the subset relation). If F contains its core, then F contains a minimal element (under the subset relation)

Definition 3.1.3 Let W be a nonempty set and $F \subseteq P(W)$. Then

- (i) F is a **filter** if F contains the unit, closed under binary intersections and supplemented. F is a proper filter if in addition F does not contain the empty set.
- (ii) F is an **ultrafilter** if F is proper filter and for each $X \subseteq W$, either $X \in F$ or $X^c \in F$.
- (iii) F is **augmented** if F contains its core and is supplemented.

Lemma 3.1.4 If F is augmented, then F is closed under arbitrary intersections. In fact, if F is augmented, then F is a filter. \otimes

The converse of this result is false, but it is not very interesting since it is easy to construct collections of sets closed under intersections that are not supplemented. What is more interesting is that there are filters that are not augmented.

Example 3.1.5

Consider $W=(-1,1)$ the real open interval and

$$F = \left\{ X \subseteq (-1,1) \mid \left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq X \text{ for some natural number } n \geq 1 \right\}.$$

As $\emptyset \notin F$ and $F \neq \emptyset$, F is consistent. By construction F is closed under supersets. Obviously, F is also closed under finite intersections; indeed, let $X_1, X_2 \in F$. Then there are $n \geq 1$ and $m \geq 1$ such that

$\left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq X_1$ and $\left(-\frac{1}{m}, \frac{1}{m}\right) \subseteq X_2$. Now either $n \leq m$ or $n > m$. If

$n = m$, we are done, since in this case

$(-\frac{1}{n}, \frac{1}{n}) = (-\frac{1}{m}, \frac{1}{m}) \subseteq X_1 \cap X_2$ and so $X_1 \cap X_2 \in F$.

Suppose $n > m$. Then $(-\frac{1}{n}, \frac{1}{n}) \subseteq (-\frac{1}{m}, \frac{1}{m})$. Hence,

$(-\frac{1}{n}, \frac{1}{n}) \subseteq X_1 \cap X_2$ and so $X_1 \cap X_2 \in F$. The case $m > n$ is similar. Thus F is a consistent filter. Now, $\bigcap F = \emptyset$ and as $\emptyset \notin F$, F is not augmented. \otimes

However, as is well-known, the situation is much better when W is finite.

Fact 3.1.6 If F is closed under binary intersections, then F is closed under finite intersections. \otimes

Corollary 3.1.7 If W is finite and F is a filter over W , then F is augmented.

Proof Since W is finite and $F \subseteq P(W)$, F is finite. F is closed under binary intersections, since it is a filter. By Fact 3.1.6, F is closed under finite intersections. In particular, F being itself a finite collection of sets from F , $\bigcap F \in F$. \otimes

3.2 Neighborhood Frames and Models

We are after a mathematical structure that can tell us, for each state, the set of necessary *propositions*. At each state we list all the sets that are considered necessary. To that end, a function $N: W \rightarrow P(P(W))$ is called a **neighborhood function**. We would like to point out that we have used a lot from ‘Eric Pacuit, staff. science. uva. nl/~ epacuit).

Definition 3.2.1 A pair $\langle W, N \rangle$ is called a **neighborhood frame**, or a neighborhood system if W a nonempty set and N is a neighborhood function.

We say that $\langle W, N \rangle$ is a filter provided for each $w \in W$, $N(w)$ is a filter. Our aim here is to explain the similarities and differences between neighborhood frames and Kripke frames as a semantics for the basic modal language. With this goal in mind, we shall clarify which neighborhood frames *correspond* to relational frames in a sense defined below. Given a relation R on a set W , define the following functions:

$$R^\rightarrow : W \rightarrow P(W) : \text{for each } w \in W, \text{ let } R^\rightarrow(w) = \{v \mid wRv\}$$

$$R^\leftarrow : P(W) \rightarrow P(W) : \text{for each } X \subseteq W, \text{ let}$$

$$R^{\leftarrow}(X) = \left\{ w \mid \exists v \in X \text{ such that } wRv \right\}$$

Definition 3.2.2 Let R be a relation on a set W and a state $w \in W$. A set $X \subseteq W$ is **R-necessary at w** if $R^{\rightarrow}(w) \subseteq X$. Let N_W^R be the set of sets that are R -necessary at w (we simply write N_W if R is clear from context). That is, $N_W^R = \left\{ X \mid R^{\rightarrow}(w) \subseteq X \right\}$. For the proofs of the following results see Eric Pacuit.

Lemma 3.2.3 Let R be a relation on W . Then for each $w \in W$, N_W is augmented. \otimes

Lemma 3.2.4 Let W be a set and $R \subseteq W \times W$. Then

- (i) If R is reflexive, then for each $w \in W$, $w \in \bigcap N_W$
- (ii) If R is transitive, then for each $w \in W$, if $X \in N_W$ then $\left\{ v \mid X \in N_V \right\} \in N_W$. \otimes

Definition 3.2.5 Let $\langle W, N \rangle$ be a neighborhood frame and $\langle W, R \rangle$ be a relational frame. We say that $\langle W, N \rangle$ and $\langle W, R \rangle$ are **equivalent** if $X \subseteq W$, $X \in N(w)$ (iff) $X \in N_W^R$.

Theorem 3.2.6 Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame. \otimes

Theorem 3.2.7 Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame. \otimes

3.3 Truth in Neighborhood Models

Recall that At denotes a set of atomic sentences or propositions.

Definition 3.3.1 Let $F = \langle W, N \rangle$ be a neighborhood frame. A **model** based on F is a triple $\langle W, N, V \rangle$, where $V : At \rightarrow 2^W$ is a valuation function.

Definition 3.3.2 Let $M = \langle W, N, V \rangle$ be a model and $w \in W$. Truth of a modal formula φ is defined inductively as follows :

1. $M, w \models p$ (iff) $w \in V(p)$, $p \in At$
2. $M, w \models \neg\varphi$ (iff) $M, w \not\models \varphi$
3. $M, w \models \varphi \wedge \Psi$ (iff) $M, w \models \varphi$ and $M, w \models \Psi$
4. $M, w \models \Box \varphi$ (iff) $(\varphi)^M \in N(w)$

$$5. M, w \models \diamond \varphi \quad (\text{iff}) \quad W - (\varphi)^M \notin N(w),$$

where $(\varphi)^M$ denotes the truth set of φ , that is,

$$(\varphi)^M = \{w \mid M, w \models \varphi\}.$$

Note that 1., 2., 3. are the same as in relational semantic; only items 4. and 5. differ from those given for relational semantics.

The following properties of the truth set will be used. Let

$N : W \rightarrow P(P(W))$ be a neighborhood function.

Define $m_N : P(W) \rightarrow P(W)$ as follows : for $X \subseteq W$,

$$m_N(X) = \{w \mid X \in N(w)\}.$$

Let $M = \langle W, N, V \rangle$ be a neighborhood model. Then we have the following result.

Proposition 3.3.3

1. $(p)^M = V(p)$ for $p \in At$
2. $(\neg\varphi)^M = W - (\varphi)^M$
3. $(\varphi \wedge \Psi)^M = (\varphi)^M \cap (\Psi)^M$
4. $(\Box\varphi)^M = m_N((\varphi)^M)$
5. $(\Diamond\varphi)^M = W - m_N(W - (\varphi)^M)$.

The proof of this result is a straightforward application of the definition of truth.

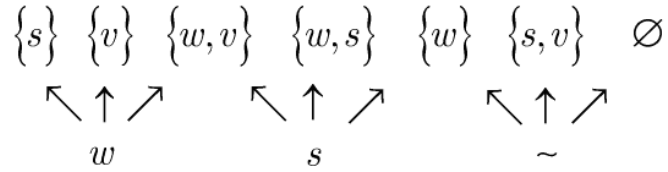
We say that φ is valid in M , denoted by $M \models \varphi$, if for each $w \in W$, $M, w \models \varphi$. In order to understand truth of a formula in a neighborhood model, we will give a detailed example.

Example 3.3.4 Let $M = \langle W, N, V \rangle$, where

- $W = \{w, s, v\}$
- $N(w) = \{\{s\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

- $V(p) = \{w, s\}$
 $V(q) = \{s, v\}$

We can picture this model as follows :



To get some feeling for this semantics, we calculate the following truth sets.

(i) $(p)^M = V(p) = \{w, s\} \in N(s)$, so we have $M, s \models \Box p$

(ii) $(\neg p)^M = \{v\} \notin N(s)$, so we have $M, s \not\models \Diamond p$

(iii) $(\Diamond p)^M = \{s, v\} \in N(v)$, so we have $M, v \models \Box \Diamond p$

(iv) $(\Box p)^M = \{s\} \in N(w)$, so we have $M, v \models \Box \Box p$

(v) $(\Box \Box p)^M = \{w\} \in N(s) \cap N(v)$, so we have

$$M, s \models \Box \Box \Box p \quad \text{and} \quad M, v \models \Box \Box \Box p$$

(vi) $(\Box \Box \Box p)^M = \{w, v\} \in N(w) \cap N(s)$, so we have

$$M, w \models \Box \Box \Box \Box p \quad \text{and} \quad M, s \models \Box \Box \Box \Box p$$

(vii) Since $(\perp)^M = \emptyset \in N(v)$, we have $M, v \models \Box \perp$.

Remarks 3.3.5

1. We have $M, w \models \Box(p \wedge q)$, but $\not\models \Box p$.
2. If we fix the valuations of p and q , it is not possible to define a relational structure such that $\Box(p \wedge q)$ is true at w but $\Box p$ is false at w . Let us see why. The condition that $\Box p$ is false at w forces w to have an accessible world in which p is false. There is only one such world, namely v , where p is false. However, if v is accessible from w , then $\Box(p \wedge q)$ will be no longer be true at w , since if p is false at v , then so is $p \wedge q$.

We already mentioned that neighborhood semantics is the generalization of Kripke semantics; in other words, formulas that are valid in Kripke models may not be so in neighborhood models.

Theorem 3.3.6 The following basic modal formulas are not valid in neighborhood models.

1. $\Box\phi \wedge \Box\Psi \rightarrow \Box(\phi \wedge \Psi)$ and $\Box(\phi \wedge \Psi) \rightarrow (\Box\phi \wedge \Box\Psi)$
2. $\Box(\phi \rightarrow \Psi) \rightarrow (\Box\phi \rightarrow \Box\Psi)$
3. $\Box\top$
4. $\Box\phi \rightarrow \phi$
5. $\Box\phi \rightarrow \Box\Box\phi$
6. $\Box\Diamond\phi \rightarrow \Diamond\Box\phi$

Proof We only consider the first two formulas; the remaining ones can be demonstrated in a similar way

1. Consider the model $M = \langle W, N, V \rangle$, where

$$W = \{w, v\}, N(w) = \{\{w\}, \{v\}\}, N(v) = \{\emptyset\}, V(p) = \{w\} \text{ and } V(q) = \{v\}.$$

Thus, $M, w \models \Box p \wedge \Box q$, but since $(p \wedge q)^M = \emptyset \notin N(w)$, $M, w \not\models \Box(p \wedge q)$.

2. Consider the model $M = \langle W, N, V \rangle$, where $W = \{w, v, s\}$,

$$N(w) = \{\{w\}, \{w, v, s\}\}, V(p) = \{w\} \text{ and } V(q) = \{w, v\}. \text{ Then } (p)^M = \{w\},$$

$$(q)^M = \{w, v\} \text{ and } (p \rightarrow q)^M = (\neg p \vee q)^M = (\neg p)^M \cup (q)^M =$$

$$[W - (p)^M] \cup (q)^M = \{v, s\} \cup \{w, v\} = \{w, s, v\}. \text{ Thus, we have}$$

$$M, w \models \Box(p \rightarrow q) \wedge \Box p, \text{ but } M, w \not\models \Box q. \otimes$$

Note that depending on the intended interpretation of the modal operators \Box and \Diamond , one might even want to work with a semantics where $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ is not valid. But in general semantics we have the following fact.

Fact 3.3.7 $\Box\phi \rightarrow \neg\Diamond\neg\phi$ is valid in all neighborhood models.

Proof Let $M = \langle W, N, V \rangle$ be an arbitrary neighborhood model. Then we have the following calculation.

$$M, w \models \Box\phi \text{ (iff) } (\phi)^M \in N(w)$$

$$\text{(iff) } W - (W - (\phi)^M) \in N(w)$$

(iff) $W - ((\neg\phi)^M) \in N(w)$

(iff) $M, w \not\models \diamond\neg\phi$

(iff) $M, w \models \neg\diamond\neg.$ \otimes

3.4 Defining Properties of Neighborhood Frames

We know that some modal formulas **define** interesting properties of a relation. In relational semantics, for example, we have shown that $F = \langle W, R \rangle$ validates $\Box\phi \rightarrow \Box\Box\phi$ (iff) R is reflexive. Similarly, modal formulas can be understood as expressing properties of neighborhood frames.

Definition 3.4.1 A modal formula ϕ defines a property P of neighborhood functions if any neighborhood frame F has property P (iff) F validates ϕ .

Remark 3.4.2

Consider the formulas $\diamond\top$ and $\Box\phi \rightarrow \diamond\phi$. On relational frames, these formulas both define the same property : that the relation is serial (i.e. every world sees a world). However, on the class of neighborhood frames, they express different properties. $\diamond\top$ expresses the fact that the empty set is not an element of the neighborhoods. The second one expresses a more interesting property about neighborhood frames.

Proposition 3.4.3 Let $F = \langle W, N \rangle$ be a neighborhood frame. Then $F \models \Box\phi \rightarrow \diamond\phi$. F is proper (i.e., if $X \in N(w)$, then $X^c \notin N(w)$).

Proof Sufficient condition is straightforward.

Necessary Condition : Assume, to the contrary, that F is not proper. Then there is a world $w \in W$ and set $X \subseteq W$ such that $X \in N(w)$ and $X^c \in N(w)$. Define a model $M = \langle W, N, V \rangle$ with $V(p) = X$. Then, by definition, $M, w \models \Box p$ and since $(\neg p)^M = X^c \in N(w)$, we have $M, w \models \neg\diamond p$ hence $M, w \not\models \Box p \rightarrow \diamond p$.

The following results and their proofs can be found in Pacuit.

Proposition 3.4.4 Let $F = \langle W, N \rangle$ be a neighborhood frame. Then

$F \models \Box\phi \wedge \Box\Psi \rightarrow \Box(\phi \wedge \Psi)$ (iff) F is closed under finite intersections. \otimes

Proposition 3.4.5 Let $F = \langle W, N \rangle$ be a neighborhood frame. Then

$F \models \Box(\phi \wedge \Psi) \rightarrow (\Box\phi \wedge \Box\Psi)$ (iff) F is closed under supersets. \otimes

Proposition 3.4.6 Let $F = \langle W, N \rangle$ be a neighborhood frame. Then $F \models \Box\top$ (iff) F contains the unit. \otimes

Theorem 3.4.7 Let $F = \langle W, N \rangle$ be a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$. Then

(i) $F \models \Box\phi \rightarrow \phi$ (iff) for each $w \in W$, $w \in \bigcap N(w)$.

(ii) $F \models \Box\phi \rightarrow \Box\Box\phi$ (iff) for each $w \in W$, if $X \in N(w)$, then $\{v \mid X \in N(v)\} \in N(w)$. \otimes

Concluding Remark

Soundness and Completeness results on neighborhood semantics are very complex with respect to the two other semantics. Thus, we will not study them in this thesis.

Modal logic has a rich model theory with respect to relational semantics [5]. Concerning monotonic neighborhood models, the situation is well-understood ([6] and see also [11]).

Further more advanced work would be to construct various models by eg. bounded morphisms, disjoint unions, bisimulations, generated submodels and the like, generalizing some classic results in modal logic (see [7]).

Finally, we would like to emphasize that much of the interest in neighborhood semantics is generated by the fact that ‘A neighborhood frame is a natural example of a Coalgebra’(see [15]).

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CURRICULUM VITAE

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